Sept 19 Math 2306 sec. 53 Fall 2018

Section 6: Linear Equations Theory and Terminology

Nonhomogeneous Equations Now we will consider the equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

where g is not the zero function. We'll continue to assume that a_n doesn't vanish and that a_i and g are continuous.

The **associated homogeneous equation** is

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0.$$

Theorem: General Solution of Nonhomogeneous Equation

Let y_p be any solution of the nonhomogeneous equation, and let y_1, $y_2, \ldots, y_n$ be any fundamental solution set of the associated homogeneous equation.

Then the general solution of the nonhomogeneous equation is

$$y = \underbrace{c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)}_{\text{This is } y_c} + y_p(x)$$

where c_1, $c_2, \ldots, c_n$ are arbitrary constants.

Note the form of the solution $y_c + y_p$!
(complementary plus particular)

Another Superposition Principle (for nonhomogeneous eqns.)

Let y_{p_1}, y_{p_2}, ..., y_{p_k} be k particular solutions to the nonhomogeneous linear equations

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g_i(x)$$

for $i = 1, \dots, k$. Assume the domain of definition for all k equations is a common interval I.

Then

$$y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_k}$$

is a particular solution of the nonhomogeneous equation

$$a_n(x)\frac{d^n y}{dx^n} + \cdots + a_0(x)y = g_1(x) + g_2(x) + \cdots + g_k(x).$$

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

We will construct the general solution by considering sub-problems.

(a) **Part 1** Verify that

$$y_{p_1} = 6 \quad \text{solves} \quad x^2y'' - 4xy' + 6y = 36.$$

$y_{p_1}' = 0$

$y_{p_1}'' = 0$

$x^2y_{p_1}'' - 4xy_{p_1}' + 6y_{p_1} \stackrel{?}{=} 36$

$x^2(0) - 4x(0) + 6(6) =$

$36 = 36$ ✓

So y_{p_1} solves this equation

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(b) **Part 2** Verify that

$$y_{p_2} = -7x \quad \text{solves} \quad x^2y'' - 4xy' + 6y = -14x.$$

$y_{p_2}' = -7$

$y_{p_2}'' = 0$

$x^2 y_{p_2}'' - 4x y_{p_2}' + 6y_{p_2} \overset{?}{=} -14x$

$x^2(0) - 4x(-7) + 6(-7x) =$

$28x - 42x =$

$-14x = -14x$ ✓

y_{p_2} does solve this equation.

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(c) **Part 3** We already know that $y_1 = x^2$ and $y_2 = x^3$ is a fundamental solution set of

$$x^2y'' - 4xy' + 6y = 0.$$

Use this along with results (a) and (b) to write the general solution of $x^2y'' - 4xy' + 6y = 36 - 14x$.

The solution is $y_c + y_p$

$y_c = c_1 y_1 + c_2 y_2 = c_1 x^2 + c_2 x^3$

By superposition, $y_p = y_{p_1} + y_{p_2} = 6 - 7x$

So the general solution is

$$y = c_1 x^2 + c_2 x^3 + 6 - 7x$$

Solve the IVP

$$x^2y'' - 4xy' + 6y = 36 - 14x, \quad y(1) = 0, \quad y'(1) = -5$$

The general solution (from the previous slides)

is $\quad y = c_1 x^2 + c_2 x^3 + 6 - 7x$

$\quad y' = 2c_1 x + 3c_2 x^2 - 7$

Applying the IC

$y(1) = 0 \Rightarrow \quad 0 = c_1 1^2 + c_2 1^3 + 6 - 7\cdot 1$

$\quad 0 = c_1 + c_2 - 1$

$y'(1) = -5 \Rightarrow \quad -5 = 2c_1 \cdot 1 + 3c_2 \cdot 1^2 - 7$

$$-5 = 2c_1 + 3c_2 - 7$$

Solve

$$c_1 + c_2 = 1 \quad \leftarrow \text{mult by 2 and subtract}$$

$$2c_1 + 3c_2 = 2$$

$$2c_1 + 2c_2 = 2$$

$$\frac{-(\ 2c_1 + 3c_2 = 2\ )}{-c_2 = 0}$$

$$c_1 = 1 - c_2 = 1 - 0 = 1$$

The solution to the IVP is

$$y = x^2 + 6 - 7x$$

Section 7: Reduction of Order

We'll focus on second order, linear, homogeneous equations. Recall that such an equation has the form

$$a_2(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_0(x)y=0.$$

Let us assume that $a_2(x)\neq 0$ on the interval of interest. We will write our equation in **standard form**

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0$$

where $P=a_1/a_2$ and $Q=a_0/a_2$.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

Recall that every fundmantal solution set will consist of two linearly independent solutions y_1 and y_2, and the general solution will have the form

$$y = c_1 y_1(x) + c_2 y_2(x).$$

Suppose we happen to know one solution $y_1(x)$. **Reduction of order** is a method for finding a second linearly independent solution $y_2(x)$ that starts with the assumption that

$$y_2(x) = u(x) y_1(x)$$

for some function $u(x)$. The method involves finding the function u.

By linear independence, we know u(x) can't be a constant function.

Example

Verify that $y_1 = e^{-x}$ is a solution of $y'' - y = 0$. Then find a second solution y_2 of the form

$$y_2(x) = u(x)y_1(x) = e^{-x}u(x).$$

Confirm that the pair y_1, y_2 is linearly independent.

Note $y_1 = e^{-x}$, $y_1' = -e^{-x}$, $y_1'' = e^{-x}$ so

$y_1'' - y_1 = e^{-x} - e^{-x} = 0 \Rightarrow y_1$ is a solution

Since y_2 is supposed to solve the ODE, we'll substitute.

$y_2 = e^{-x}u$

$y_2' = e^{-x}u' - e^{-x}u$

$$y_2'' = e^{-x}u'' - e^{-x}u' - e^{-x}u' + e^{-x}u$$

$$= e^{-x}u'' - 2e^{-x}u' + e^{-x}u$$

$$y_2'' - y_2 = 0$$

$$e^{-x}u'' - 2e^{-x}u' + e^{-x}u - e^{-x}u = 0$$

$$e^{-x}u'' - 2e^{-x}u' = 0$$

$$e^{-x}(u'' - 2u') = 0$$

$$u'' - 2u' = 0$$

Let $w = u'$, then $w' = u''$. The equation is

$$w' - 2w = 0$$

So we can solve this as separable or linear.

$$\frac{dw}{dx} = 2w \qquad \text{separate}$$

$$\int \frac{1}{w}\, dw = \int 2\, dx$$

$$\ln|w| = 2x$$

Taking the added constant to be zero

$$|w| = e^{2x}$$

assume $w > 0$

$w = e^{2x}$ $\quad w = u'$ so integrate

$u = \int e^{2x}\,dx = \frac{1}{2}e^{2x}$ so

$y_2 = u y_1 = \frac{1}{2}e^{2x} \cdot e^{-x} = \frac{1}{2}e^{x}$

Our pair of functions is $y_1 = e^{-x}$, $y_2 = \frac{1}{2}e^{x}$

We'll use the Wronskian to show they are linearly independent.

$$W(y_1, y_2)(x) = \begin{vmatrix} e^{-x} & \frac{1}{2}e^{x} \\ -e^{-x} & \frac{1}{2}e^{x} \end{vmatrix}$$

$$= e^{-x}\left(\frac{1}{2}e^{x}\right) - \left(-e^{-x}\right)\left(\frac{1}{2}e^{x}\right)$$

$$= \frac{1}{2} - \left(-\frac{1}{2}\right) = \frac{1}{2} + \frac{1}{2} = 1$$

We have a fundamental solution set.

The general solution to $y'' - y = 0$

is

$$\boxed{y = c_1 e^{-x} + c_2 e^{x}}$$