Section 6: Linear Equations Theory and Terminology

We first consider the homogeneous equation

\[ a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = 0 \]

and assume that each \( a_i \) is continuous and \( a_n \) is never zero on the interval of interest.

**Theorem:** If \( y_1, y_2, \ldots, y_k \) are all solutions of this homogeneous equation on an interval \( I \), then the linear combination

\[ y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x) \]

is also a solution on \( I \) for any choice of constants \( c_1, \ldots, c_k \).

This is called the **principle of superposition**.
We Defined Linear Dependence/Independence

**Definition:** A set of functions \( f_1(x), f_2(x), \ldots, f_n(x) \) are said to be **linearly dependent** on an interval \( I \) if there exists a set of constants \( c_1, c_2, \ldots, c_n \) with at least one of them being nonzero such that

\[
c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \text{for all} \quad x \text{ in } I.
\]

A set of functions that is not linearly dependent on \( I \) is said to be **linearly independent** on \( I \).
Definition of Wronskian

Let \( f_1, f_2, \ldots, f_n \) possess at least \( n - 1 \) continuous derivatives on an interval \( I \). The **Wronskian** of this set of functions is the determinant

\[
W(f_1, f_2, \ldots, f_n)(x) = \begin{vmatrix}
  f_1 & f_2 & \cdots & f_n \\
  f'_1 & f'_2 & \cdots & f'_n \\
  \vdots & \vdots & \ddots & \vdots \\
  f^{(n-1)}_1 & f^{(n-1)}_2 & \cdots & f^{(n-1)}_n \\
\end{vmatrix}.
\]

(Note that, in general, this Wronskian is a function of the independent variable \( x \).)
Recall Some Determinants

If $A$ is a $2 \times 2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then its determinant

$$\det(A) = ad - bc.$$ 

If $A$ is a $3 \times 3$ matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then its determinant

$$\det(A) = a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}.$$
Determine the Wronskian of the Functions

\[ f_1(x) = x^2, \quad f_2(x) = 4x, \quad f_3(x) = x - x^2 \]

\[ 3 \text{ functions} \Rightarrow 3 \times 3 \]

\[
W(f_1, f_2, f_3)(x) = \begin{vmatrix}
    x^2 & 4x & x - x^2 \\
    2x & 4 & 1 - 2x \\
    2 & 0 & -2
\end{vmatrix}
\]

\[
= x^2 \begin{vmatrix} 4 & 1 - 2x \\ 0 & -2 \end{vmatrix} - 4x \begin{vmatrix} 2x & 1 - 2x \\ 2 & -2 \end{vmatrix} + (x - x^2) \begin{vmatrix} 2x & 4 \\ 2 & 0 \end{vmatrix}
\]
\[ = x^2 \left( 4 \cdot (-2) - 0 \cdot (1 - 2x) \right) - 4x \left( 2x \cdot (-2) - 2 \cdot (1 - 2x) \right) \]
\[ + (x - x^2) \left( 2x \cdot (0) - 2 \cdot 4 \right) \]
\[ = -8x^2 - 4x (-4x - 2 + 4x) - 8x + 8x^2 \]
\[ = -8x^2 + 8x - 8x + 8x^2 \]
\[ = 0 \]
5. \[ W(f_1, f_2, f_3)(x) = 0 \]
Theorem (a test for linear independence)

Let $f_1, f_2, \ldots, f_n$ be $n - 1$ times continuously differentiable on an interval $I$. If there exists $x_0$ in $I$ such that $W(f_1, f_2, \ldots, f_n)(x_0) \neq 0$, then the functions are linearly independent on $I$.

If $y_1, y_2, \ldots, y_n$ are $n$ solutions of the linear homogeneous $n^{th}$ order equation on an interval $I$, then the solutions are linearly independent on $I$ if and only if $W(y_1, y_2, \ldots, y_n)(x) \neq 0$ for each $x$ in $I$.

\footnote{For solutions of one linear homogeneous ODE, the Wronskian is either always zero or is never zero.}
Determine if the functions are linearly dependent or independent:

\[ y_1 = e^x, \quad y_2 = e^{-x} \quad I = (-\infty, \infty) \]

We'll use the Wronskian

\[ y_1' = e^x, \quad y_2' = -e^x \quad \text{2 fns.} \quad \Rightarrow \quad \text{2x2 matrix} \]

\[
W(y_1, y_2)(x) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^x \end{vmatrix} = e^x(-e^x) - e^{-x}(e^x) = -e^x - e^{-x} = -1 - 1 = -2
\]

\[ W(y_1, y_2)(x) = -2 \neq 0 \]
Hence $y_1, y_2$ are linearly independent.
Fundamental Solution Set

We’re still considering this equation

\[ a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = 0 \]

with the assumptions \( a_n(x) \neq 0 \) and \( a_i(x) \) are continuous on \( I \).

**Definition:** A set of functions \( y_1, y_2, \ldots, y_n \) is a fundamental solution set of the \( n^{th} \) order homogeneous equation provided they

(i) are solutions of the equation,

(ii) there are \( n \) of them, and

(iii) they are linearly independent.

**Theorem:** Under the assumed conditions, the equation has a fundamental solution set.
General Solution of $n^{th}$ order Linear Homogeneous Equation

Let $y_1, y_2, \ldots, y_n$ be a fundamental solution set of the $n^{th}$ order linear homogeneous equation. Then the \textbf{general solution} of the equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.
Example

Verify that \( y_1 = e^x \) and \( y_2 = e^{-x} \) form a fundamental solution set of the ODE

\[ y'' - y = 0 \quad \text{on} \quad (-\infty, \infty), \]

and determine the general solution.

Let’s verify that they are solutions.

\[
\begin{align*}
y_1 &= e^x, \quad y_1' = e^x, \quad y_1'' = e^x \\
y_2 &= e^{-x}, \quad y_2' = -e^{-x}, \quad y_2'' = e^{-x}
\end{align*}
\]

\[
y_1'' - y_1 = e^x - e^x = 0 \quad \text{so \( y_1 \) solves the ODE}
\]

\[
y_2'' - y_2 = e^{-x} - e^{-x} = 0 \quad \text{so \( y_2 \) solves the ODE}
\]
The equation is 2nd order, \( n=2 \) and we have 2 solutions, \( y_1 \) and \( y_2 \). We took the Wronskian and found they are linearly independent. So \( y_1, y_2 \) constitutes a fundamental solution set.

\[
\text{The general solution is } \quad y = c_1 y_1 + c_2 y_2
\]

\[
= c_1 e^\frac{x}{2} + c_2 e^{-\frac{x}{2}}
\]
Nonhomogeneous Equations

Now we will consider the equation

\[ a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x) \]

where \( g \) is not the zero function. We’ll continue to assume that \( a_n \) doesn’t vanish and that \( a_i \) and \( g \) are continuous.

The associated homogeneous equation is

\[ a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = 0. \]
Theorem: General Solution of Nonhomogeneous Equation

Let \( y_p \) be any solution of the nonhomogeneous equation, and let \( y_1, y_2, \ldots, y_n \) be any fundamental solution set of the associated homogeneous equation.

Then the general solution of the nonhomogeneous equation is

\[
y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p(x)
\]

where \( c_1, c_2, \ldots, c_n \) are arbitrary constants.

Note the form of the solution \( y_c + y_p \! \) (complementary plus particular)
Another Superposition Principle (for nonhomogeneous eqns.)

Let \( y_{p_1}, y_{p_2}, \ldots, y_{p_k} \) be \( k \) particular solutions to the nonhomogeneous linear equations

\[
a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = g_i(x)
\]

for \( i = 1, \ldots, k \). Assume the domain of definition for all \( k \) equations is a common interval \( I \).

Then

\[
y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_k}
\]

is a particular solution of the nonhomogeneous equation

\[
a_n(x) \frac{d^n y}{dx^n} + \cdots + a_0(x)y = g_1(x) + g_2(x) + \cdots + g_k(x).
\]
Example \( x^2 y'' - 4xy' + 6y = 36 - 14x \) \( I = (0, \infty) \)

(a) Verify that

\[ y_{p_1} = 6 \] solves \( x^2 y'' - 4xy' + 6y = 36 \).

Substitute \( y_{p_1} = 6, \quad y_{p_1} = 0, \quad y_{p_1} = 0 \)

\[ x^2 y_{p_1}'' - 4xy_{p_1}' + 6y_{p_1} = \]

\[ x^2 (6) - 4x (0) + 6 (6) = 36 \]
Example \( x^2y'' - 4xy' + 6y = 36 - 14x \)

(b) Verify that 

\[ y_{p_2} = -7x \text{ solves } x^2y'' - 4xy' + 6y = -14x. \]

Substitute \( y_{p_2} = -7x, \quad y_{p_2}' = -7, \quad y_{p_2}'' = 0 \)

\[ x^2y_{p_2}'' - 4xy_{p_2}' + 6y_{p_2} = \]

\[ x^2(0) - 4x(-7) + 6(-7x) = \]

\[ 28x - 42x = -14x \]
Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(c) It is readily shown that $y_1 = x^2$ and $y_2 = x^3$ is a fundamental solution set of

$$x^2y'' - 4xy' + 6y = 0.$$  

Use this along with results (a) and (b) to write the general solution of $x^2y'' - 4xy' + 6y = 36 - 14x$.

$$y_c = c_1y_1 + c_2y_2 = c_1x^2 + c_2x^3$$

$$y_p = y_{p1} + y_{p2} = 6 - 7x$$

The general solution to the nonhomogeneous equation is

$$y = c_1x^2 + c_2x^3 + 6 - 7x$$
Section 7: Reduction of Order

We’ll focus on second order, linear, homogeneous equations. Recall that such an equation has the form

\[ a_2(x) \frac{d^2y}{dx^2} + a_1(x) \frac{dy}{dx} + a_0(x)y = 0. \]

Let us assume that \( a_2(x) \neq 0 \) on the interval of interest. We will write our equation in standard form

\[ \frac{d^2y}{dx^2} + P(x) \frac{dy}{dx} + Q(x)y = 0 \]

where \( P = a_1/a_2 \) and \( Q = a_0/a_2 \).
\[
\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0
\]

Recall that every fundamental solution set will consist of two linearly independent solutions \(y_1\) and \(y_2\), and the general solution will have the form

\[
y = c_1 y_1(x) + c_2 y_2(x).
\]

Suppose we happen to know one solution \(y_1(x)\). Reduction of order is a method for finding a second linearly independent solution \(y_2(x)\) that starts with the assumption that

\[
y_2(x) = u(x)y_1(x)
\]

for some function \(u(x)\). The method involves finding the function \(u\).
Example

Verify that $y_1 = e^{-x}$ is a solution of $y'' - y = 0$. Then find a second solution $y_2$ of the form

$$y_2(x) = u(x)y_1(x) = e^{-x}u(x).$$

Confirm that the pair $y_1, y_2$ is linearly independent.

Find $y_2'$ and $y_2''$

$$y_2' = e^{-x}u'(x) - e^{-x}u(x)$$

$$y_2'' = e^{-x}u''(x) - e^{-x}u'(x) - e^{-x}u'(x) + e^{-x}u(x)$$

$$= e^{-x}u''(x) - 2e^{-x}u'(x) + e^{-x}u(x)$$
We require \( y_2'' - y_2 = 0 \)

\[
y_2'' - y_2 = e^x u''(x) - 2 e^x u'(x) + e^x u(x) - e^x u(x) = 0
\]

\[
\Rightarrow e^x u''(x) - 2 e^x u'(x) = 0
\]

\[
e^x (u''(x) - 2u'(x)) = 0
\]

\[
u''(x) - 2u'(x) = 0
\]

Let \( w = u' \), then \( w' = u'' \). The equation

U must solve this equation.
\[ w' - 2w = 0 \quad \text{Separable} \]

\[ \frac{dw}{dx} = 2w \]

\[ \frac{1}{w} \frac{dw}{dx} = 2 \]

\[ \int \frac{1}{w} \, dw = \int 2 \, dx \]

\[ \ln|w| = 2x \quad \text{Taking the } +C \text{ to be zero.} \]
Exponentiate, let's assume \( w > 0 \)

\[ e^{\ln w} = e^x \]

\[ w = e^x \]

Now, \( w = u \) so \( u = \int wdx = \int e^x dx \)

\[ u = \frac{1}{2} e^{2x} \]

Finally, \( y_2 = uy_1 = \frac{1}{2} e^{2x} \cdot e^{-x} = \frac{1}{2} e^x \)
A fundamental solution set is
\[ y_1 = e^{-x}, \quad y_2 = \frac{1}{2} e^x \]

The general solution is
\[ y = c_1 y_1 + c_2 y_2 \]
\[ y = c_1 e^{-x} + c_2 \left( \frac{1}{2} e^x \right) \]
\[ y = k_1 e^{-x} + k_2 e^x \]