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Section 6: Linear Equations Theory and Terminology

We first consider the homogeneous equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

and assume that each a_i is continuous and a_n is never zero on the interval of interest.

Theorem: If $y_1, y_2, \ldots, y_k$ are all solutions of this homogeneous equation on an interval I, then the *linear combination*

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x)$$

is also a solution on I for any choice of constants $c_1, \ldots, c_k$.

This is called the **principle of superposition.**

We Defined Linear Dependence/Independence

Definition: A set of functions $f_1(x), f_2(x), \dots, f_n(x)$ are said to be **linearly dependent** on an interval I if there exists a set of constants $c_1, c_2, \dots, c_n$ with at least one of them being nonzero such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \text{for all} \quad x \text{ in } I.$$

A set of functions that is not linearly dependent on I is said to be **linearly independent** on I.

Definition of Wronskian

Let f_1, $f_2, \ldots, f_n$ posses at least $n-1$ continuous derivatives on an interval I. The **Wronskian** of this set of functions is the determinant

$$W(f_1, f_2, \ldots, f_n)(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

(Note that, in general, this Wronskian is a function of the independent variable x.)

Recall Some Determinants

If A is a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then its determinant

$$\det(A) = ad - bc.$$

If A is a 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then its determinant

$$\det(A) = a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Determine the Wronskian of the Functions

$$f_1(x) = x^2, \quad f_2(x) = 4x, \quad f_3(x) = x - x^2$$

3 functions $\Rightarrow$ 3×3

$$W(f_1, f_2, f_3)(x) = \begin{vmatrix} x^2 & 4x & x-x^2 \\ 2x & 4 & 1-2x \\ 2 & 0 & -2 \end{vmatrix}$$

$$= x^2 \begin{vmatrix} 4 & 1-2x \\ 0 & -2 \end{vmatrix} - 4x \begin{vmatrix} 2x & 1-2x \\ 2 & -2 \end{vmatrix} + (x-x^2) \begin{vmatrix} 2x & 4 \\ 2 & 0 \end{vmatrix}$$

$$= x^2\left(4\cdot(-2) - 0\cdot(1-2x)\right) - 4x\left(2x(-2) - 2(1-2x)\right)$$

$$+ (x-x^2)\left(2x(0) - 2\cdot 4\right)$$

$$= -8x^2 - 4x\left(-4x - 2 + 4x\right) - 8x + 8x^2$$

$$= -8x^2 + 8x - 8x + 8x^2$$

$$= 0$$

So $W(f_1, f_2, f_3)(x) = 0$

Theorem (a test for linear independence)

Let $f_1, f_2, \ldots, f_n$ be $n-1$ times continuously differentiable on an interval I. If there exists x_0 in I such that $W(f_1, f_2, \ldots, f_n)(x_0) \neq 0$, then the functions are **linearly independent** on I.

If $y_1, y_2, \ldots, y_n$ are n solutions of the linear homogeneous n^{th} order equation on an interval I, then the solutions are **linearly independent** on I if and only if $W(y_1, y_2, \ldots, y_n)(x) \neq 0$ for[1] each x in I.

[1] For solutions of one linear homogeneous ODE, the Wronskian is either always zero or is never zero.

Determine if the functions are linearly dependent or independent:

$y_1 = e^x, \quad y_2 = e^{-x} \quad I = (-\infty, \infty)$ We'll use the wronskian

$y_1' = e^x, \quad y_2' = -e^{-x}$ 2 fnct. $\Rightarrow$ 2x2 matrix

$$W(y_1, y_2)(x) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = e^x(-e^{-x}) - e^x(e^{-x})$$

$$= -e^0 - e^0 = -1 - 1 = -2$$

$$W(y_1, y_2)(x) = -2 \neq 0$$

Hence y_1, y_2 are linearly independent.

Fundamental Solution Set

We're still considering this equation

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=0$$

with the assumptions $a_n(x)\neq 0$ and $a_i(x)$ are continuous on I.

Definition: A set of functions $y_1, y_2, \ldots, y_n$ is a **fundamental solution set** of the n^{th} order homogeneous equation provided they

(i) are solutions of the equation,

(ii) there are n of them, and

(iii) they are linearly independent.

Theorem: Under the assumed conditions, the equation has a fundamental solution set.

General Solution of n^{th} order Linear Homogeneous Equation

Let $y_1, y_2, \ldots, y_n$ be a fundamental solution set of the n^{th} order linear homogeneous equation. Then the **general solution** of the equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Example

Verify that $y_1 = e^x$ and $y_2 = e^{-x}$ form a fundamental solution set of the ODE

$$y'' - y = 0 \quad \text{on} \quad (-\infty, \infty),$$

and determine the general solution.

Let's verify that they are solutions.

$y_1 = e^x$, $y_1' = e^x$, $y_1'' = e^x$

$y_1'' - y_1 = e^x - e^x = 0$ so y_1 solves the ODE

$y_2 = e^{-x}$, $y_2' = -e^{-x}$, $y_2'' = e^{-x}$

$y_2'' - y_2 = e^{-x} - e^{-x} = 0$ So y_2 solves the ODE

The equation is 2^{nd} order, $n=2$ and we have 2 solutions y_1 and y_2.

We took the Wronskian and found they are linearly independent. So y_1, y_2 constitutes a fundamental solution set.

The general solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 e^{x} + c_2 e^{-x}$$

Nonhomogeneous Equations

Now we will consider the equation

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

where g is not the zero function. We'll continue to assume that a_n doesn't vanish and that a_i and g are continuous.

The **associated homogeneous equation** is

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=0.$$

Theorem: General Solution of Nonhomogeneous Equation

Let y_p be any solution of the nonhomogeneous equation, and let y_1, $y_2, \ldots, y_n$ be any fundamental solution set of the associated homogeneous equation.

Then the general solution of the nonhomogeneous equation is

$$y = \underbrace{c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)}_{y_c} + y_p(x)$$

where c_1, $c_2, \ldots, c_n$ are arbitrary constants.

Note the form of the solution $y_c + y_p$!
(complementary plus particular)

Another Superposition Principle (for nonhomogeneous eqns.)

Let y_{p_1}, y_{p_2}, ..., y_{p_k} be k particular solutions to the nonhomogeneous linear equations

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g_i(x)$$

for $i = 1, \ldots, k$. Assume the domain of definition for all k equations is a common interval I.

Then

$$y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_k}$$

is a particular solution of the nonhomogeneous equation

$$a_n(x)\frac{d^ny}{dx^n} + \cdots + a_0(x)y = g_1(x) + g_2(x) + \cdots + g_k(x).$$

Example $x^2y'' - 4xy' + 6y = 36 - 14x$ $\quad I = (0, \infty)$

(a) Verify that

$$y_{p_1} = 6 \quad \text{solves} \quad x^2y'' - 4xy' + 6y = 36.$$

Substitute $\quad y_{p_1} = 6, \quad y_{p_1}' = 0, \quad y_{p_1}'' = 0$

$$x^2 y_{p_1}'' - 4x y_{p_1}' + 6y_{p_1} =$$

$$x^2(0) - 4x(0) + 6(6) = 36$$

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(b) Verify that

$$y_{p_2} = -7x \quad \text{solves} \quad x^2y'' - 4xy' + 6y = -14x.$$

Substitute $\quad y_{p_2} = -7x, \quad y_{p_2}' = -7, \quad y_{p_2}'' = 0$

$$x^2 y_{p_2}'' - 4x\, y_{p_2}' + 6\, y_{p_2} =$$

$$x^2(0) - 4x(-7) + 6(-7x) =$$

$$28x - 42x = -14x$$

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(c) It is readily shown that $y_1 = x^2$ and $y_2 = x^3$ is a fundamental solution set of

$$x^2y'' - 4xy' + 6y = 0. \leftarrow \text{assoc. homogeneous eqn.}$$

Use this along with results (a) and (b) to write the general solution of $x^2y'' - 4xy' + 6y = 36 - 14x$.

$$y_c = c_1y_1 + c_2y_2 = c_1x^2 + c_2x^3$$

$$y_p = y_{p_1} + y_{p_2} = 6 - 7x$$

The gen. solution to the nonhomogeneous equation is

$$\boxed{y = c_1x^2 + c_2x^3 + 6 - 7x}$$

Section 7: Reduction of Order

We'll focus on second order, linear, homogeneous equations. Recall that such an equation has the form

$$a_2(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_0(x)y=0.$$

Let us assume that $a_2(x)\neq 0$ on the interval of interest. We will write our equation in **standard form**

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0$$

where $P=a_1/a_2$ and $Q=a_0/a_2$.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

Recall that every fundmantal solution set will consist of two linearly independent solutions y_1 and y_2, and the general solution will have the form

$$y = c_1 y_1(x) + c_2 y_2(x).$$

Suppose we happen to know one solution $y_1(x)$. **Reduction of order** is a method for finding a second linearly independent solution $y_2(x)$ that starts with the assumption that

$$y_2(x) = u(x) y_1(x)$$

for some function $u(x)$. The method involves finding the function u.

Example

Verify that $y_1 = e^{-x}$ is a solution of $y'' - y = 0$. Then find a second solution y_2 of the form

$$y_2(x) = u(x)y_1(x) = e^{-x}u(x).$$

Confirm that the pair y_1, y_2 is linearly independent.

Find y_2' and y_2''

$$y_2' = e^{-x}u'(x) - e^{-x}u(x)$$

$$y_2'' = e^{-x}u''(x) - e^{-x}u'(x) - e^{-x}u'(x) + e^{-x}u(x)$$

$$= e^{-x}u''(x) - 2e^{-x}u'(x) + e^{-x}u(x)$$

We require $\quad y_2'' - y_2 = 0$

$$y_2'' - y_2 = e^{-x}u''(x) - 2e^{-x}u'(x) + e^{-x}\cancel{u(x)} - e^{-x}\cancel{u(x)} = 0$$

$$\Rightarrow \quad e^{-x}u''(x) - 2e^{-x}u'(x) = 0$$

$$e^{-x}\left(u''(x) - 2u'(x)\right) = 0$$

$$u''(x) - 2u'(x) = 0$$

u must solve this equation

Let $w = u'$, then $w' = u''$. The equation in w is

$$w' - 2w = 0$$ Separable

$$\frac{dw}{dx} = 2w$$

$$\frac{1}{w}\frac{dw}{dx} = 2$$

$$\int \frac{1}{w}\,dw = \int 2\,dx$$

$$\ln|w| = 2x$$ Taking the $+C$ to be zero.

Exponentiate, let's assume $w>0$

$$e^{\ln w} = e^{2x}$$

$$w = e^{2x}$$

Now, $w=u'$ so $u = \int w\,dx = \int e^{2x}\,dx$

$$u = \frac{1}{2}e^{2x}$$

Finally, $y_2 = uy_1 = \frac{1}{2}e^{2x}\cdot e^{-x} = \frac{1}{2}e^{x}$

A fundamental solution set is

$$y_1 = e^{-x}, \quad y_2 = \frac{1}{2}e^{x}$$

The general solution is

$$y = c_1 y_1 + c_2 y_2$$

$$y = c_1 e^{-x} + c_2\left(\frac{1}{2}e^{x}\right)$$

$$\boxed{y = k_1 e^{-x} + k_2 e^{x}}$$