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Section 6: Linear Equations Theory and Terminology

We first consider the homogeneous equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

and assume that each a_i is continuous and a_n is never zero on the interval of interest.

Theorem: If $y_1, y_2, \ldots, y_k$ are all solutions of this homogeneous equation on an interval I, then the *linear combination*

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x)$$

is also a solution on I for any choice of constants $c_1, \ldots, c_k$.

This is called the **principle of superposition**.

We Defined Linear Dependence/Independence

Definition: A set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ are said to be **linearly dependent** on an interval I if there exists a set of constants $c_1, c_2, \ldots, c_n$ with at least one of them being nonzero such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \quad \text{for all} \quad x \text{ in } I.$$

A set of functions that is not linearly dependent on I is said to be **linearly independent** on I.

Definition of Wronskian

Let $f_1, f_2, \ldots, f_n$ posses at least $n-1$ continuous derivatives on an interval I. The **Wronskian** of this set of functions is the determinant

$$W(f_1, f_2, \ldots, f_n)(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

(Note that, in general, this Wronskian is a function of the independent variable x.)

Recall Some Determinants

If A is a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then its determinant

$$\det(A) = ad - bc.$$

If A is a 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then its determinant

$$\det(A) = a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Determine the Wronskian of the Functions

$$f_1(x) = x^2, \quad f_2(x) = 4x, \quad f_3(x) = x - x^2$$

3 functions $\Rightarrow$ 3×3 matrix

$$W(f_1, f_2, f_3)(x) = \begin{vmatrix} x^2 & 4x & x-x^2 \\ 2x & 4 & 1-2x \\ 2 & 0 & -2 \end{vmatrix}$$

$$= x^2 \begin{vmatrix} 4 & 1-2x \\ 0 & -2 \end{vmatrix} - 4x \begin{vmatrix} 2x & 1-2x \\ 2 & -2 \end{vmatrix} + (x-x^2) \begin{vmatrix} 2x & 4 \\ 2 & 0 \end{vmatrix}$$

$$= x^2 \left(4(-2) - 0 \cdot (1-2x) \right) - 4x \left(2x(-2) - 2(1-2x) \right)$$

$$+ (x - x^2) \left(2x \cdot 0 - 2 \cdot 4 \right)$$

$$= -8x^2 - 4x \left(-4x - 2 + 4x \right) + (x - x^2)(-8)$$

$$= -8x^2 + 8x - 8x + 8x^2$$

$$= 0$$

5. $\quad W(f_1, f_2, f_3)(x) = 0$

Theorem (a test for linear independence)

Let $f_1, f_2, \ldots, f_n$ be $n-1$ times continuously differentiable on an interval I. If there exists x_0 in I such that $W(f_1, f_2, \ldots, f_n)(x_0) \neq 0$, then the functions are **linearly independent** on I.

If $y_1, y_2, \ldots, y_n$ are n solutions of the linear homogeneous n^{th} order equation on an interval I, then the solutions are **linearly independent** on I if and only if $W(y_1, y_2, \ldots, y_n)(x) \neq 0$ for[1] each x in I.

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[1] For solutions of one linear homogeneous ODE, the Wronskian is either always zero or is never zero.

Determine if the functions are linearly dependent or independent:

$y_1 = e^x, \quad y_2 = e^{-x} \quad I = (-\infty, \infty)$

We can use the Wronskian

$y_1' = e^x, \quad y_2' = -e^{-x}$

2 fnctions $\Rightarrow$ 2×2

$$W(y_1, y_2)(x) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}$$

$$= e^x(-e^{-x}) - e^x(e^{-x})$$

$$= -e^0 - e^0 = -1 - 1 = -2$$

$$W(y_1, y_2)(x) = -2 \neq 0$$

y_1, y_2 are linearly independent.

Fundamental Solution Set

We're still considering this equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

with the assumptions $a_n(x) \neq 0$ and $a_i(x)$ are continuous on I.

Definition: A set of functions $y_1, y_2, \ldots, y_n$ is a **fundamental solution set** of the nth order homogeneous equation provided they
  (i) are solutions of the equation,
  (ii) there are n of them, and
  (iii) they are linearly independent.

Theorem: Under the assumed conditions, the equation has a fundamental solution set.

General Solution of nth order Linear Homogeneous Equation

Let $y_1, y_2, \ldots, y_n$ be a fundamental solution set of the nth order linear homogeneous equation. Then the **general solution** of the equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Example

Verify that $y_1 = e^x$ and $y_2 = e^{-x}$ form a fundamental solution set of the ODE

$$y'' - y = 0 \quad \text{on} \quad (-\infty, \infty),$$

and determine the general solution.

Let's verify that y_1, y_2 solve the ODE.

$y_1 = e^x$ $y_1'' - y_1 = e^x - e^x = 0$ y_1 solves the ODE

$y_1' = e^x$

$y_1'' = e^x$

$y_2 = e^{-x}$ $y_2'' - y_2 = e^{-x} - e^{-x} = 0$ y_2 solves the ODE

$y_2' = -e^{-x}$

$y_2'' = e^{-x}$

We have 2 solutions for the 2nd order
equation.

We found the Wronskian $W(y_1, y_2)(x) = -2$
so they are linearly independent.

They constitute a fundamental Solution set.

The general solution is

$$y = c_1 y_1 + c_2 y_2$$

$$y = c_1 e^{x} + c_2 e^{-x}.$$

Nonhomogeneous Equations

Now we will consider the equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

where g is not the zero function. We'll continue to assume that a_n doesn't vanish and that a_i and g are continuous.

The **associated homogeneous equation** is

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0.$$

Theorem: General Solution of Nonhomogeneous Equation

Let y_p be any solution of the nonhomogeneous equation, and let y_1, $y_2, \ldots, y_n$ be any fundamental solution set of the associated homogeneous equation.

Then the general solution of the nonhomogeneous equation is

$$y = \underbrace{c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)}_{y_c} + y_p(x)$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Note the form of the solution $y_c + y_p$!
(complementary plus particular)

Another Superposition Principle (for nonhomogeneous eqns.)

Let $y_{p_1}, y_{p_2}, \ldots, y_{p_k}$ be k particular solutions to the nonhomogeneous linear equations

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g_i(x)$$

for $i = 1, \ldots, k$. Assume the domain of definition for all k equations is a common interval I.

Then

$$y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_k}$$

is a particular solution of the nonhomogeneous equation

$$a_n(x)\frac{d^n y}{dx^n} + \cdots + a_0(x)y = g_1(x) + g_2(x) + \cdots + g_k(x).$$

Example $x^2 y'' - 4xy' + 6y = 36 - 14x$ $I = (0, \infty)$

(a) Verify that

$$y_{p_1} = 6 \quad \text{solves} \quad x^2 y'' - 4xy' + 6y = 36.$$

Substitute $y_{p_1} = 6$, $y_{p_1}' = 0$, $y_{p_1}'' = 0$

$$x^2 y_{p_1}'' - 4x y_{p_1}' + 6 y_{p_1} =$$

$$x^2(0) - 4x(0) + 6(6) = 36 \qquad y_{p_1} \text{ solves this eqn.}$$

Example $x^2 y'' - 4xy' + 6y = 36 - 14x$

(b) Verify that

$$y_{p_2} = -7x \quad \text{solves} \quad x^2 y'' - 4xy' + 6y = -14x.$$

$y_{p_2} = -7x \qquad y_{p_2}' = -7, \quad y_{p_2}'' = 0$

$x^2 y_{p_2}'' - 4x y_{p_2}' + 6 y_{p_2} =$

$x^2 (0) - 4x(-7) + 6(-7x) =$

$28x - 42x \quad = -14x \qquad$ y_{p_2} solves this ODE

Example $x^2 y'' - 4xy' + 6y = 36 - 14x$

(c) It is readily shown that $y_1 = x^2$ and $y_2 = x^3$ is a fundamental solution set of

$$x^2 y'' - 4xy' + 6y = 0. \quad \leftarrow \text{assoc. homogeneous eqn.}$$

Use this along with results (a) and (b) to write the general solution of $x^2 y'' - 4xy' + 6y = 36 - 14x$.

$$y_c = C_1 y_1 + C_2 y_2 = C_1 x^2 + C_2 x^3$$

$$y_p = y_{p_1} + y_{p_2} = 6 - 7x$$

The general solution to the non homogeneous equation

$$y = C_1 x^2 + C_2 x^3 + 6 - 7x$$

$$y_c + y_p$$

Section 7: Reduction of Order

We'll focus on second order, linear, homogeneous equations. Recall that such an equation has the form

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0.$$

Let us assume that $a_2(x) \neq 0$ on the interval of interest. We will write our equation in **standard form**

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

where $P = a_1/a_2$ and $Q = a_0/a_2$.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

Recall that every fundmantal solution set will consist of two linearly independent solutions y_1 and y_2, and the general solution will have the form

$$y = c_1 y_1(x) + c_2 y_2(x).$$

Suppose we happen to know one solution $y_1(x)$. **Reduction of order** is a method for finding a second linearly independent solution $y_2(x)$ that starts with the assumption that

$$y_2(x) = u(x)y_1(x)$$

for some function $u(x)$. The method involves finding the function u.

Example

Verify that $y_1 = e^{-x}$ is a solution of $y'' - y = 0$. Then find a second solution y_2 of the form

$$y_2(x) = u(x)y_1(x) = e^{-x}u(x).$$

Confirm that the pair y_1, y_2 is linearly independent.

We'll sub y_2 into the ODE

$$y_2 = e^{-x} u(x)$$

$$y_2' = e^{-x} u'(x) - e^{-x} u(x)$$

$$y_2'' = e^{-x} u''(x) - e^{-x} u'(x) - e^{-x} u'(x) + e^{-x} u(x)$$

$$= e^{-x} u''(x) - 2e^{-x} u'(x) + e^{-x} u(x)$$

$$y_2'' - y_2 = e^{-x} u''(x) - 2e^{-x} u'(x) + e^{-x} u(x) - e^{-x} u(x) = 0$$

$$e^{-x} u''(x) - 2e^{-x} u'(x) = 0$$

$$e^{-x} \left(u''(x) - 2u'(x) \right) = 0$$

$$u''(x) - 2u'(x) = 0 \qquad \text{u must solve this ODE}$$

Let $w = u'$, then $w' = u''$

The equation for w is

$$w' - 2w = 0 \qquad \text{1st order Separable}$$

I'll assume $w > 0$

$$\frac{dw}{dx} = 2w \implies \frac{1}{w}\frac{dw}{dx} = 2$$

$$\int \frac{1}{w}\,dw = \int 2\,dx$$

$$\ln w = 2x$$

We'll take the $+C$
to be zero.

$$w = e^{2x}$$

$$w = u' \implies u = \int w\,dx$$

$$u = \int e^{2x} \, dx = \frac{1}{2} e^{2x} \qquad \text{again taking} \\ \qquad\qquad\qquad\qquad\qquad +C \text{ as zero.}$$

$$y_2 = u y_1 \qquad \text{so} \qquad y_2 = \frac{1}{2} e^{2x} \cdot e^{-x} = \frac{1}{2} e^{x}$$

The general solution is

$$y = C_1 y_1 + C_2 y_2 = C_1 e^{-x} + C_2 \left(\frac{1}{2} e^{x} \right)$$

i.e.

$$y = k_1 e^{-x} + k_2 e^{x}$$