September 21 MATH 1113 sec. 51 Fall 2018

Section 4.5: Rational Functions

Vertical asymptotes correspond to unbounded $f(x)$ values. What about when the independent variable x becomes unbounded?

Definition: The horizontal line $y = b$ is a horizontal asymptote to the function f if the graph of $f(x)$ hugs the line $y = b$ as x becomes unbounded. Symbolically

$$f(x) \to b \quad \text{as} \quad x \to \infty \quad \text{or as} \quad x \to -\infty.$$

Horizontal Asymptotes

Some remarks about horizontal asymptotes:

- A rational function can have **at most** one horizontal asymptote. Meaning if $f(x) \to b$ as $x \to \infty$ then it is also true that $f(x) \to b$ as $x \to -\infty$.
- A rational function **MAY** cross a horizontal asymptote. (It might not, but it's possible.)
- A horizontal asymptote is a **line**! So $y = 3$ *could* be a horizontal asymptote, but 3 **CANNOT** be a horizontal asymptote.

Finding Horizontal Asymptotes

Let $f(x) = \frac{p(x)}{q(x)}$ be a rational function[1].

$$f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0}$$

Let n be the degree of p, m be the degree of q and a_n and b_m the leading coefficients for p and q respectively. There are three cases:

(I) If $n < m$, then $y = 0$ is the horizontal asymptote to f.

(II) If $n = m$, then $y = \frac{a_n}{b_m}$ is the horizontal asymptote to f.

(III) If $n > m$, then f does not have a horizontal asymptote.

[1] The results are the same whether f is in lowest terms or not.

A word on the relative degrees test

We determine whether a horizontal asymptote exists as well as its equation by comparing the degrees of the numberator and denominator.

While this seems like a gimmick, it can be shown rigorously using tools in Calculus!

Example

Determine the equation of any horizontal asymptote to the rational function f. If one exists, does the graph cross it?

(a) $f(x) = \dfrac{3x^2+7x-2}{x^2+2x}$ $= \dfrac{P(x)}{Q(x)}$ $n=2$ $m=2$

so $n=m$

There is a horizontal asymptote.

Here $a_n = a_2 = 3$

$b_m = b_2 = 1$

So the asymptote is $y = \frac{3}{1} \Rightarrow \boxed{y=3}$

Does f cross the asymptote?

We set $f(x)$ equal to y (where $y=3$).

Set $f(x)=3$ try to solve for x.

$$\frac{3x^2+7x-2}{x^2+2x}=3 \implies 3x^2+7x-2=3(x^2+2x)=3x^2+6x$$

$$7x-2=6x$$

$$x=2$$

Check to be sure and to see if 2 is in the domain.

$$f(2)=\frac{3(2)^2+7(2)-2}{2^2+2\cdot2}=\frac{12+14-2}{8}=\frac{24}{8}=3$$

The graph crosses @ $(2,3)$.

(b) $f(x) = \dfrac{2x^4 + 9x^2 + 4}{5x^6 + 3x^3 - 2x^2 + 1} = \dfrac{p(x)}{q(x)}$ $n = 4$ $m = 6$

$n < m$

There is a horizontal asymptote $y = 0$.

Does the graph cross?

Set $f(x) = y$ where $y = 0$

$$\frac{2x^4 + 9x^2 + 4}{5x^6 + 3x^3 - 2x^2 + 1} = 0$$

$$\Rightarrow \quad \underbrace{2x^4 + 9x^2 + 4}_{\text{this side is greater than or equal to } 4} = 0$$

$\Rightarrow$ no solutions.

The graph does not cross the asymptote.

Question

Does the graph of $f(x) = \dfrac{x^2 + 2x}{x+1}$ have a horizontal asymptote?

(a) Yes, its equation is $y = 1$.

$n = 2$, $m = 1$

(b) Yes, its equation is $y = 0$.

$n > m$

(c) Yes, its equation is $y = x + 1$.

(d) No, it does not.

Oblique Asymptotes

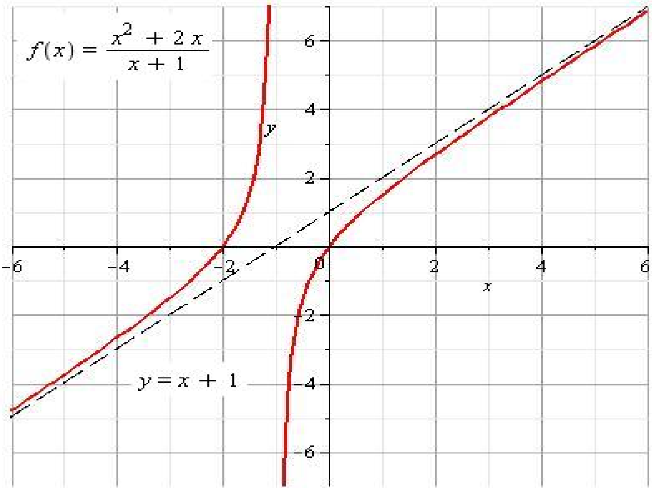


Figure: Graph of $f(x) = \frac{x^2+2x}{x+1}$ together with the line $y = x + 1$.

Oblique Asymptote

From Case III: If $n = m + 1$ (i.e. n is *exactly* one more than m), then f has an **oblique** asymptote

$$y = mx + b.$$

To find the asymptote, we perform long division to write

$$f(x) = \frac{p(x)}{q(x)} = mx + b + \frac{r(x)}{q(x)}.$$

The degree of the remainder polynomial r will be less than the degree of q.

If $n \geq m + 2$, then f has no horizontal and no oblique asymptotes.

Example

Determine the oblique asymptote for the rational function. Does the graph cross it?

(a) $f(x) = \dfrac{x^2+2x}{x+1}$

Divide x^2+2x by $x+1$

$$\begin{array}{r|l} & x+1 \quad \leftarrow \text{quotient} \\ x+1 & x^2+2x+0 \\ & -(x^2+x) \\ \hline & x+0 \\ & -(x+1) \\ \hline & -1 \quad \leftarrow \text{remainder} \end{array}$$

$\Rightarrow f(x) = x+1 + \dfrac{-1}{x+1}$

$\Downarrow$

form

$mx+b + \dfrac{r(x)}{q(x)}$

The oblique asymptote is $y = x+1$.

Does it cross? Set $f(x)$ equal to y.

$$f(x) = x+1 - \frac{1}{x+1} \qquad y = x+1$$

$$\cancel{x+1} - \frac{1}{x+1} = \cancel{x+1}$$

$$\frac{-1}{x+1} = 0$$

$$\Rightarrow \quad -1 = 0 \quad \text{always false}$$

$f(x) = y$ has no solution.

The graph does not cross.

Question

Use long division to find the equation of the oblique asymptote to the graph of $g(x) = \dfrac{2x^3 - x^2 - x + 3}{x^2 - 1}$. (Hint: Write the divisor as $x^2 + 0x - 1$ to align correctly when doing division.)

The equation of the oblique asymptote is

(a) $y = 2x$

(b) $y = 2x - 1$

(c) $y = 2x + 1$

(d) $y = 2x + 2$

Graphing Rational Functions

We can obtain a graph of a rational function in steps and by identifying key features. These include:

- the domain of the function,
- putting it in lowest terms,
- finding any vertical asymptotes
- finding a horizontal or an oblique asymptote if one exists (determine if the graph crosses)
- find the y-intercept if 0 is in the domain (i.e. find $f(0)$)
- find any x-intercepts (i.e. solve $p(x) = 0$)
- identify behavior near asymptotes (plot at least one point between each intercept and vert. asymptote)

Plot $f(x) = \frac{x+4}{x^2-3}$ [2]

Determine the domain, and put f into lowest terms.

$$f(x) = \frac{x+4}{(x-\sqrt{3})(x+\sqrt{3})}$$

$g(x) = (x-\sqrt{3})(x+\sqrt{3})$ so $g(x) = 0$ if $x = \pm\sqrt{3}$.

The domain is $\{x \mid x \neq \pm\sqrt{3}\}$.

f is in lowest terms

[2] We'll plot on the graph a few slides down.

$f(x) = \frac{x+4}{x^2-3} = \frac{x+4}{(x-\sqrt{3})(x+\sqrt{3})}$

Find the equation(s) of any vertical asymptotes.

There are two V. asymptotes

$x = \sqrt{3}$ and $x = -\sqrt{3}$

$f(x) = \frac{x+4}{x^2-3}$

Identify any horizontal or oblique asymptote, and identify any points at which the graph crosses.

Here $n = 1$ $m = 2$

$n < m$

There is a horizontal asymptote

$\underline{y = 0}$

Does it cross? $\frac{x+4}{x^2-3} = 0 \Rightarrow x+4 = 0$

$x = -4$

It crosses at $x = -4$. The point

$(-4, 0)$ is on the graph.

$f(x) = \frac{x+4}{x^2-3}$

Identify the points of any x and y intercepts.

y-intercept : 0 is in the domain.

$$f(0) = \frac{0+4}{0^2-3} = \frac{-4}{3}$$

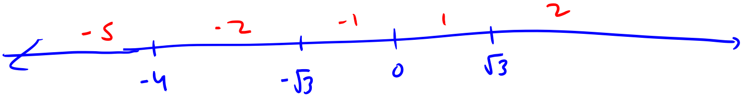
$\left(0, \frac{-4}{3}\right)$

The x-intercept is $(-4,0)$ (found earlier).

$f(x) = \frac{x+4}{x^2-3}$

Identify points on the graphs—in particular points between intercepts and vertical asymptotes.

Divide the $\mathbb{R}$ line



pick points in between, and find points on the graph

$$f(-5) = \frac{-5+4}{(-5)^2-3} = \frac{-1}{22}$$

$$f(-2) = \frac{-2+4}{(-2)^2-3} = \frac{2}{1} = 2$$

$f(x) = \frac{x+4}{x^2-3}$

Identify points on the graphs—in particular points between intercepts and vertical asymptotes.

$$f(-1) = \frac{-1+4}{(-1)^2-3} = \frac{3}{-2}$$

$$f(1) = \frac{1+4}{1^2-3} = \frac{-5}{2}$$

$$f(2) = \frac{2+4}{2^2-3} = \frac{6}{1} = 6$$

$f(x) = \frac{x+4}{x^2-3}$

Intercepts $(-4, 0)$ and $(0, \frac{-4}{3})$

Interval	$(-\infty, -4)$	$(-4, -\sqrt{3})$	$(-\sqrt{3}, 0)$	$(0, \sqrt{3})$	$(\sqrt{3}, \infty)$	
test pt c	-5	-2	-1	1	2	
$f(c)$	$\frac{-1}{22}$	2	$\frac{-3}{2}$	$\frac{-5}{2}$	6	
sign	$-$	$+$	$-$	$-$	$+$	

$$f(x) = \frac{x+4}{x^2-3}$$