September 21 MATH 1113 sec. 52 Fall 2018

Section 4.5: Rational Functions

Vertical asymptotes correspond to unbounded $f(x)$ values. What about when the independent variable x becomes unbounded?

Definition: The horizontal line $y = b$ is a horizontal asymptote to the function f if the graph of $f(x)$ hugs the line $y = b$ as x becomes unbounded. Symbolically

$$f(x) \to b \quad \text{as} \quad x \to \infty \quad \text{or as} \quad x \to -\infty.$$

Horizontal Asymptotes

Some remarks about horizontal asymptotes:

- A rational function can have **at most** one horizontal asymptote. Meaning if $f(x) \to b$ as $x \to \infty$ then it is also true that $f(x) \to b$ as $x \to -\infty$.

- A rational function **MAY** cross a horizontal asymptote. (It might not, but it's possible.)

- A horizontal asymptote is a **line**! So $y = 3$ *could* be a horizontal asymptote, but 3 **CANNOT** be a horizontal asymptote.

Finding Horizontal Asymptotes

Let $f(x) = \frac{p(x)}{q(x)}$ be a rational function[1].

$$f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0}$$

Let n be the degree of p, m be the degree of q and a_n and b_m the leading coefficients for p and q respectively. There are three cases:

(I) If $n < m$, then $y = 0$ is the horizontal asymptote to f.

(II) If $n = m$, then $y = \frac{a_n}{b_m}$ is the horizontal asymptote to f.

(III) If $n > m$, then f does not have a horizontal asymptote.

[1] The results are the same whether f is in lowest terms or not.

A word on the relative degrees test

We determine whether a horizontal asymptote exists as well as its equation by comparing the degrees of the numberator and denominator.

While this seems like a gimmick, it can be shown rigorously using tools in Calculus!

Example

Determine the equation of any horizontal asymptote to the rational function f. If one exists, does the graph cross it?

(a) $f(x) = \dfrac{3x^2 + 7x - 2}{x^2 + 2x} = \dfrac{p(x)}{q(x)}$

$n = 2 \qquad m = 2$

$n = m$ case

There is a horizontal asymptote

$y = \dfrac{a_n}{b_m} = \dfrac{a_2}{b_2} \qquad a_2 = 3, \; b_2 = 1$

The asymptote is $y = \frac{3}{1}$ i.e. $y = 3$.

Does f cross the asymptote? To answer this we set $f(x)$ equal to the $y=3$ y and solve for x.

Set $f(x) = 3$

$$(x^2+2x)\ \frac{3x^2+7x-2}{x^2+2x} = 3\ (x^2+2x) \Rightarrow$$

$$\cancel{3x^2}+7x-2 = 3(x^2+2x) = \cancel{3x^2}+6x$$

$$7x-2 = 6x \quad \Rightarrow \quad x = 2$$

Double check: $\quad f(2) = \dfrac{3(2)^2+7(2)-2}{2^2+2(2)} = \dfrac{24}{8} = 3$

The graph crosses the asymptote @ $(2,3)$.

(b) $f(x) = \dfrac{2x^4 + 9x^2 + 4}{5x^6 + 3x^3 - 2x^2 + 1} = \dfrac{p(x)}{q(x)}$

$n = 4 \quad m = 6$

$n < m$

There is a H. asymptote.

The asymptote is $y = 0$.

Does it cross? Set $f(x)$ equal to y.

$$\frac{2x^4 + 9x^2 + 4}{5x^6 + 3x^3 - 2x^2 + 1} = 0$$

$$2x^4 + 9x^2 + 4 = 0$$

$2x^4+9x^2+4$ is greater than or equal to 4 for all x. There are no solutions to $2x^4+9x^2+4=0$.

f does not cross the asymptote.

Question

Does the graph of $f(x) = \dfrac{x^2 + 2x}{x + 1}$ have a horizontal asymptote?

$n = 2$

$m = 1 \qquad n > m$

(a) Yes, its equation is $y = 1$.

(b) Yes, its equation is $y = 0$.

(c) Yes, its equation is $y = x + 1$.

(d) No, it does not.

Oblique Asymptotes

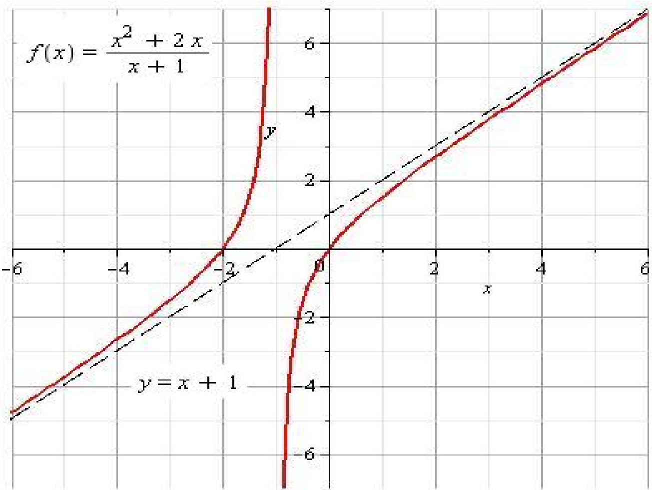

Figure: Graph of $f(x) = \frac{x^2+2x}{x+1}$ together with the line $y = x + 1$.

Oblique Asymptote

From Case III: If $n = m + 1$ (i.e. n is *exactly* one more than m), then f has an **oblique** asymptote

$$y = mx + b.$$

To find the asymptote, we perform long division to write

$$f(x) = \frac{p(x)}{q(x)} = mx + b + \frac{r(x)}{q(x)}.$$

The degree of the remainder polynomial r will be less than the degree of q.

If $n \geq m + 2$, then f has no horizontal and no oblique asymptotes.

Example

Determine the oblique asymptote for the rational function. Does the graph cross it?

(a) $f(x) = \dfrac{x^2+2x}{x+1}$

Divide $p(x)$ by $q(x)$

$$\begin{array}{r|l} & x+1 \quad \leftarrow \text{quotient} \\ \hline x+1 & x^2+2x+0 \\ & -(x^2+x) \\ \hline & \quad x+0 \\ & \quad -(x+1) \\ \hline & \quad\quad -1 \quad \leftarrow \text{remainder} \end{array}$$

$\Rightarrow f(x) = x+1 + \dfrac{-1}{x+1}$

$\downarrow \quad \downarrow$ form

$mx+b + \dfrac{r(x)}{q(x)}$

The oblique asymptote is $y = x+1$.

Does it cross? Set $f(x)$ equal to y.

$$\cancel{x+1} - \frac{1}{x+1} = \cancel{x+1}$$

$$\frac{-1}{x+1} = 0 \implies -1 = 0 \quad \text{always false}$$

f does not cross the oblique asymptote.

Question

Use long division to find the equation of the oblique asymptote to the graph of $g(x) = \dfrac{2x^3 - x^2 - x + 3}{x^2 - 1}$. (Hint: Write the divisor as $x^2 + 0x - 1$ to align correctly when doing division.)

The equation of the oblique asymptote is

(a) $y = 2x$

(b) $y = 2x - 1$

(c) $y = 2x + 1$

(d) $y = 2x + 2$