Sept. 21 Math 1190 sec. 51 Fall 2016

Section 2.5: The Derivative of the Trigonometric Functions

We wish to arrive at derivative rules for each of the six trigonometric functions.

The first two are

$$\frac{d}{dx}\sin(x) = \cos(x) \quad \text{and} \quad \frac{d}{dx}\cos(x) = -\sin(x)$$

I proved the first in class, the second is left as an **assigned** homework exercise.

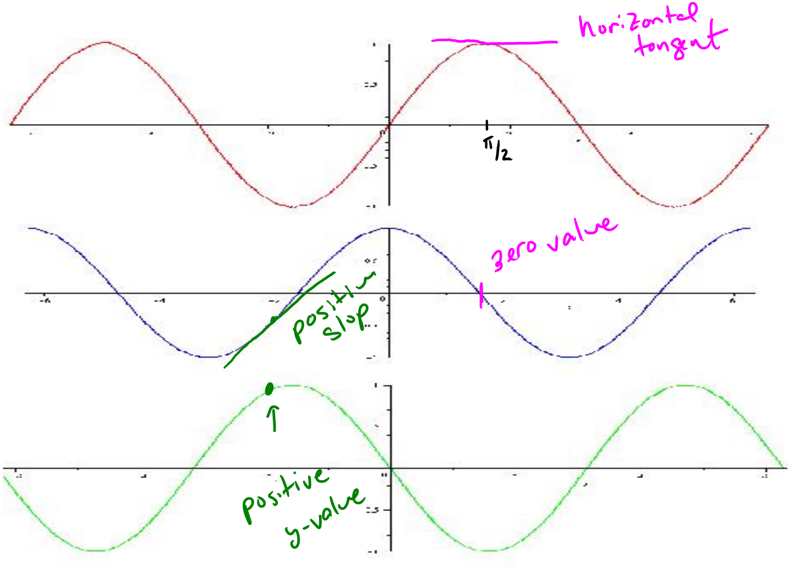

Figure: Graphs of $y = \sin x$, $y = \cos x$, $y = -\sin x$ (from top to bottom).

Examples: Evaluate the derivative.

(a) $\dfrac{d}{dx}(\sin x + 4\cos x) = \dfrac{d}{dx}\sin x + 4\dfrac{d}{dx}\cos x$

$= \cos x + 4(-\sin x)$

$= \cos x - 4\sin x$

(b) $\dfrac{d}{d\theta}\theta^4 \sin\theta = \left(\dfrac{d}{d\theta}\theta^4\right)\sin\theta + \theta^4\dfrac{d}{d\theta}\sin\theta$ (product rule)

$= 4\theta^3\sin\theta + \theta^4\cos\theta$

Question

Find $f'(x)$ where $f(x) = \sin x \cos x$

$f'(x) = \cos x \cos x + \sin x(-\sin x)$

(a) $f'(x) = \cos^2 x - \sin^2 x$

(b) $f'(x) = 1$

(c) $f'(x) = -\cos x \sin x$

(d) $f'(x) = 2 \sin x \cos x$

Use the fact that $\tan x = \sin x / \cos x$ to determine the derivative rule for the tangent.

Quotient Rule

$$\frac{d}{dx}\frac{f}{g} = \frac{f'g - fg'}{g^2}$$

$$\frac{d}{dx}\tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{\left(\frac{d}{dx}\sin x\right)\cos x - \sin x\left(\frac{d}{dx}\cos x\right)}{(\cos x)^2}$$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

$$\frac{d}{dx} \tan x = \sec^2 x$$

Six Trig Function Derivatives

$$\frac{d}{dx}\sin x = \cos x, \qquad \frac{d}{dx}\cos x = -\sin x,$$

$$\frac{d}{dx}\tan x = \sec^2 x, \qquad \frac{d}{dx}\cot x = -\csc^2 x,$$

$$\frac{d}{dx}\sec x = \sec x \tan x, \qquad \frac{d}{dx}\csc x = -\csc x \cot x$$

Example

Find $\frac{dg}{dt}$.

$g(t) = e^t - 2\cot t$

$$g'(t) = e^t - 2(-\csc^2 t)$$

$$= e^t + 2\csc^2 t$$

Example

Find the equation of the line tangent to the graph of $y = \sec x$ at the point $(\pi/3, 2)$.

We need the slope. $m_{tan} = y'\left(\frac{\pi}{3}\right)$

$y'(x) = \sec x \tan x$

so $m_{tan} = \sec\frac{\pi}{3}\tan\frac{\pi}{3} = 2\sqrt{3}$

$y - y_0 = m(x - x_0)$

$y - 2 = 2\sqrt{3}\left(x - \frac{\pi}{3}\right)$

$$\boxed{y = 2\sqrt{3}\,x - \frac{2\sqrt{3}\pi}{3} + 2}$$

Question

Quotient rule

$$\frac{d}{dx}\frac{f}{g} = \frac{f'g - fg'}{g^2}$$

Find $f'(x)$ where $f(x) = \dfrac{x^2+2}{\tan x}$

(a) $f'(x) = \dfrac{2x}{\sec^2 x}$

(b) $f'(x) = \dfrac{2x\tan x - (x^2+2)\sec^2 x}{\tan^2 x}$

(c) $f'(x) = \dfrac{2x\tan x + (x^2+2)\sec^2 x}{\tan^2 x}$

(d) $f'(x) = \dfrac{2x - (x^2+2)\sec^2 x}{\tan x}$

Section 3.1: The Chain Rule

Suppose we wish to find the derivative of $f(x) = (x^2 + 2)^2$.

$f(x) = x^4 + 4x^2 + 4$ expand the square

Now $f'(x) = 4x^3 + 8x$

Now suppose we want to differentiate $g(x) = (x^2 + 2)^{10}$. How about $F(x) = \sqrt{x^2 + 2}$?

we could expand g, but it would be very tedious.

we can't take the derivative of F with any rules so far.

Example of Compositions

Find functions $f(u)$ and $g(x)$ such that

$$F(x) = \sqrt{x^2 + 2} = (f \circ g)(x).$$

To determine how compositions are nested, we can look at the order of operations used to evaluate the function.

outside function

$$(f \circ g)(x) = f(g(x))$$

inside function

To evaluate $F(x)$

(1) square x and add 2, then

(2) take the square root.

$g(x) = x^2 + 2$ and $f(u) = \sqrt{u}$

Note then that if $u = g(x)$

$$(f \circ g)(x) = f(g(x)) = f(x^2+2) = \sqrt{x^2+2}$$

as expected.

Example of Compositions

Find functions $f(u)$ and $g(x)$ such that

$$F(x) = \cos\left(\frac{\pi x}{2}\right) = (f \circ g)(x).$$

To evaluate F, (1) multiply by $\frac{\pi}{2}$, then

(2) take the cosine

$g(x) = \frac{\pi}{2}x$ and $f(u) = \cos u$

$(f \circ g)(x) = f(g(x)) = f\left(\frac{\pi}{2}x\right) = \cos\left(\frac{\pi}{2}x\right)$

Theorem: Chain Rule

Suppose g is differentiable at x and f is differentiable at $g(x)$. Then the composite function

$$F = f \circ g$$

is differentiable at x and

$$\frac{d}{dx} F(x) = \frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$

In Liebniz notation: if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}.$$

Example

Determine any inside and outside functions and find the derivative.

(a) $F(x) = \sin^2 x$ $= (\sin x)^2$

(1) take sine

(2) Square

Chain rule

$F(x) = f(g(x))$ so

$F'(x) = f'(g(x)) \cdot g'(x)$

$g(x) = \sin x$

$f(u) = u^2$

$g'(x) = \cos x$

$f'(u) = 2u$

$F'(x) = 2u \cos x$

$= 2 \sin x \cos x$

(b) $F(x) = e^{x^4 - 5x^2 + 1}$

$u = g(x) = x^4 - 5x^2 + 1$

$f(u) = e^u$

$F'(x) = f'(g(x))\, g'(x)$

$g'(x) = 4x^3 - 10x$

$= e^u (4x^3 - 10x)$

$f'(u) = e^u$

$= e^{x^4 - 5x^2 + 1} (4x^3 - 10x)$

(c) $G(\theta) = \cos\left(\frac{\pi\theta}{2} - \frac{\pi}{4}\right)$

$u = g(\theta) = \frac{\pi}{2}\theta - \frac{\pi}{4}$

$f(u) = \cos u$

$G'(\theta) = f'(g(\theta))\, g'(\theta)$

$g'(\theta) = \frac{\pi}{2}$

$= -\sin u \left(\frac{\pi}{2}\right)$

$f'(u) = -\sin u$

$= -\frac{\pi}{2} \sin\left(\frac{\pi}{2}\theta - \frac{\pi}{4}\right)$

Question

Find $h'(x)$ where $h(x) = \tan(2x)$

$g(x) = 2x$ $g'(x) = 2$

$f(u) = \tan u$ $f'(u) = \sec^2 u$

so

$h'(x) = \sec^2 u \cdot 2$

$= 2\sec^2(2x)$

(a) $h'(x) = 2\tan(2x)$

(b) $h'(x) = \sec^2(2x)$

(c) $h'(x) = 2\sec^2(2x)$

(d) $h'(x) = 2\tan(2x)\sec^2(2x)$

The power rule with the chain rule

If $u = g(x)$ is a differentiable function and n is any integer, then

$$\frac{d}{dx}\, u^n = n u^{n-1} \frac{du}{dx}.$$

Evaluate: $\dfrac{d}{dx}\left(\dfrac{x-1}{2x+2}\right)^7$

$u = \dfrac{x-1}{2x+2} = g(x)$

$$\frac{du}{dx} = \frac{1(2x+2) - (x-1)(2)}{(2x+2)^2} = \frac{2x+2-2x+2}{(2x+2)^2}$$

$$= \frac{4}{(2x+2)^2} = \frac{4}{2^2(x+1)^2} = \frac{1}{(x+1)^2}$$

$$\frac{d}{dx}\left(\frac{x-1}{2x+2}\right)^7 = 7u^6 \cdot \frac{du}{dx}$$

$$= 7\left(\frac{x-1}{2x+2}\right)^6 \left(\frac{1}{(x+1)^2}\right)$$

$$= \frac{7(x-1)^6}{(x+1)^2(2x+2)^6}$$

Example

Find the first and second derivatives of $y = \cot(2t)$ with respect to t.

$u = 2t$, $\quad f(u) = \cot u \qquad y = f(u)$

$\frac{du}{dt} = 2 \qquad \frac{df}{du} = -\csc^2 u$

$$\frac{dy}{dt} = \frac{dy}{du}\frac{du}{dt} = -\csc^2 u \cdot (2) = -2\csc^2(2t)$$

$$\frac{dy}{dt} = -2\big(\csc(2t)\big)^2$$

$u = g(t) = 2t$, $\quad v = h(u) = \csc u$

$$f(v) = -2v^2$$

$$\frac{du}{dt} = 2 \quad , \quad \frac{dv}{du} = -\csc u \cot u \quad , \quad \frac{df}{dv} = -4v$$

$$\frac{d^2y}{dt^2} = \frac{df}{dv}\,\frac{dv}{dt} = \frac{df}{dv}\left(\frac{dv}{du}\,\frac{du}{dt}\right)$$

$$\frac{d^2y}{dt^2} = -4v\,(-\csc u \cot u)\cdot 2$$

$$= -4 \csc u\,\left(-\csc(2t)\cot(2t)\right)\cdot 2$$

$$= 8\csc(2t)\csc(2t)\cot(2t)$$

$$= 8\csc^2(2t)\cot(2t)$$