

Section 4.2 Reduction of Order

Recall that a second order, homogeneous, linear equation in standard form looks like

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0.$$

Any fundamental solution set will necessarily consist of two (y_1, y_2) linearly independent solutions.

Herein, we will assume that one such solution $y_1(x)$ is known. We seek the second in the form

$$y_2(x) = u(x)y_1(x)$$

for some function $u(x)$.

Generalization

Consider the equation **in standard form** with one known solution.
Determine a second linearly independent solution.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \quad y_1(x) \text{ --- is known.}$$

We started this by setting $y_2 = uy_1$ and found the derivatives

$$y'_2 = u'y_1 + uy'_1$$

$$y''_2 = u''y_1 + 2u'y'_1 + uy''_1$$

$$y''_2 + P(x)y'_2 + Q(x)y_2 = 0$$

$$u''y_1 + 2u'y_1' + uy_1'' + P(x)(u'y_1 + uy_1') + Q(x)uy_1 = 0$$

$$y_1 u'' + (2y_1' + P(x)y_1)u' + \underbrace{(y_1'' + P(x)y_1' + Q(x)y_1)}_{=0} u = 0$$

y_1 solves the homogeneous equation

$$\text{i.e. } y_1'' + P(x)y_1' + Q(x)y_1 = 0$$

$$\Rightarrow y_1 u'' + (2y_1' + P(x)y_1)u' = 0 \quad \text{1st order in } u'$$

$$\text{Let } w = u' \text{ so } w' = u''$$

$$y_1 w' + (2y_1' + p(x)y_1)w = 0$$

$$w' + \left(\frac{2y_1'}{y_1} + p(x) \right) w = 0$$

$$\frac{dw}{dx} = - \left(\frac{2y_1'}{y_1} + p(x) \right) w$$

$$\frac{1}{w} \frac{dw}{dx} dx = - \left(\frac{2y_1'}{y_1} + p(x) \right) dx$$

$$\int \frac{1}{w} dw = - \int 2 \frac{dy_1}{y_1} - \int p(x) dx$$

$$\ln w = -2 \ln y_1 - \int p(x) dx$$

$$e^{\ln w} = e^{-2 \ln y_1 - \int p(x) dx}$$

$$w = e^{\ln y_1^{-2}} \cdot e^{-\int p(x) dx} = y_1^{-2} e^{-\int p(x) dx}$$

Since $w = u'$, $u = \int w dx \Rightarrow$

$$u = \int \frac{e^{-\int p(t) dt}}{(y_1(x))^2} dx$$

and $y_2(x) = u(x)y_1(x) = y_1(x) \int \frac{e^{-\int p(t) dt}}{(y_1(x))^2} dx$

We found

$$u(x) = \int \frac{e^{-\int p(x)dx}}{(y_1(x))^2} dx$$

and the

second solution

$$y_2(x) = y_1(x) u(x)$$

Reduction of Order Formula

For the second order, homogeneous equation **in standard form**

$$y'' + P(x)y' + Q(x)y = 0$$

with one known solution y_1 , a second linearly independent solution y_2 is given by

$$y_2 = y_1(x) \int \frac{e^{-\int P(x) dx}}{(y_1(x))^2} dx$$

Example

Find the general solution of the ODE given one known solution

$$x^2 y'' - 3xy' + 4y = 0, \quad y_1 = x^2 \quad \text{take } x > 0$$

Standard form $y'' - \frac{3}{x} y' + \frac{4}{x^2} y = 0$

$$P(x) = -\frac{3}{x}, \text{ so } - \int P(x) dx = - \int \frac{-3}{x} dx = 3 \ln x = \ln(x^3)$$

$$\text{so } e^{- \int P(x) dx} = e^{\ln x^3} = x^3$$

$$u(x) = \int \frac{e^{- \int P(x) dx}}{(y_1(x))^2} dx = \int \frac{x^3}{(x^2)^2} dx$$

$$= \int \frac{x^3}{x^4} dx = \int \frac{1}{x} dx = \ln x$$

$$y_2(x) = u(x)y_1(x) = x^2 \ln x$$

The general solution to the ODE is

$$y = C_1 y_1 + C_2 y_2 \Rightarrow$$

$$y = C_1 x^2 + C_2 x^2 \ln x$$

As a side note, let's see that y_1, y_2 are linearly independent.

$$W(y_1, y_2)(x) = \begin{vmatrix} x^2 & x^2 \ln x \\ 2x & 2x \ln x + x \end{vmatrix}$$
$$= x^2(2x \ln x + x) - 2x(x^2 \ln x)$$
$$= 2x^3 \ln x + x^3 - 2x^3 \ln x = x^3 \neq 0$$

They are linearly independent.

Example

Find the solution of the IVP where one solution of the ODE is given.

$$y'' + 4y' + 4y = 0 \quad y_1 = e^{-2x}, \quad y(0) = 1, \quad y'(0) = -2$$

Get y_2 using reduction of order.

$$P(x) = 4 \quad \text{so} \quad -\int P(x) dx = -\int 4 dx = -4x$$

$$e^{-\int P(x) dx} = e^{-4x}$$

$$u(x) = \int \frac{e^{-\int P(x) dx}}{(y_1(x))^2} dx = \int \frac{e^{-4x}}{(e^{-2x})^2} dx = \int \frac{e^{-4x}}{e^{-4x}} dx$$

$$u(x) = \int dx = x, \quad y_2(x) = u(x)y_1(x) = xe^{-2x}$$

The general solution to the ODE is

$$y = C_1 e^{-2x} + C_2 x e^{-2x}$$

Apply the I.C. $y(0) = 1$ $y'(0) = -2$

$$y' = -2C_1 e^{-2x} - 2C_2 x e^{-2x} + C_2 e^{-2x}$$

$$y(0) = C_1 e^0 + C_2 \cdot 0 e^0 = 1 \Rightarrow C_1 = 1$$

$$y'(0) = -2 \cdot 1 e^0 - 2C_2 \cdot 0 e^0 + C_2 e^0 = -2$$

$$-2 + C_2 = -2 \Rightarrow C_2 = 0$$

The solution to the IJP is

$$y = e^{-2x}.$$

Section 4.3: Homogeneous Equations with Constant Coefficients

We consider a second order, linear, homogeneous equation with constant coefficients

$$a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0.$$

Question: What sort of function y could be expected to satisfy

$$y'' = \text{constant} \times y' + \text{constant} \times y?$$

We look for solutions of the form $y = e^{mx}$ with m constant.

$$ay'' + by' + cy = 0 \quad \text{assume } y = e^{mx}, m \text{-constant}$$

$$y = e^{mx}, \quad y' = me^{mx}, \quad y'' = m^2 e^{mx}$$

$$a(m^2 e^{mx}) + b(me^{mx}) + c e^{mx} = 0$$

$$e^{mx} (am^2 + bm + c) = 0$$

To be true, it must be that

$$am^2 + bm + c = 0$$

m must satisfy this quadratic

equation called the

Characteristic equation for the

ODE

Auxiliary a.k.a. Characteristic Equation

$$am^2 + bm + c = 0$$

There are three cases:

- I $b^2 - 4ac > 0$ and there are two distinct real roots $m_1 \neq m_2$
- II $b^2 - 4ac = 0$ and there is one repeated real root $m_1 = m_2 = m$
- III $b^2 - 4ac < 0$ and there are two roots that are complex conjugates
 $m_{1,2} = \alpha \pm i\beta$

Case I: Two distinct real roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac > 0$$

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} \quad \text{where} \quad m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Show that $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$ are linearly independent.

$$\begin{aligned} W(y_1, y_2)(x) &= \begin{vmatrix} e^{m_1 x} & e^{m_2 x} \\ M_1 e^{m_1 x} & M_2 e^{m_2 x} \end{vmatrix} = e^{m_1 x} (M_2 e^{m_2 x}) - M_1 e^{m_1 x} (e^{m_2 x}) \\ &= M_2 e^{(m_1+m_2)x} - M_1 e^{(m_1+m_2)x} = (M_2 - M_1) e^{(m_1+m_2)x} \neq 0 \end{aligned}$$

Because $M_2 \neq M_1$, i.e., $M_2 - M_1 \neq 0$

Example

Find the general solution of the ODE

$$y'' - 2y' - 2 = 0$$

Characteristic equation: $m^2 - 2m - 2 = 0$

Roots: $m^2 - 2m + 1 = 2 + 1$

$$(m-1)^2 = 3 \Rightarrow m-1 = \pm\sqrt{3}$$

$$\Rightarrow m = 1 \pm \sqrt{3} \quad 2 \text{ distinct roots}$$

$$m_1 = 1 + \sqrt{3}, \quad m_2 = 1 - \sqrt{3}$$

$$y_1 = e^{m_1 x} = e^{(1+\sqrt{3})x}, \quad y_2 = e^{m_2 x} = e^{(1-\sqrt{3})x}$$

General Solution

$$y = c_1 e^{(1+\sqrt{3})x} + c_2 e^{(1-\sqrt{3})x}$$