Sept 21 Math 2306 sec. 53 Fall 2018

Section 7: Reduction of Order

We'll focus on second order, linear, homogeneous equations. Recall that such an equation has the form

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0.$$

Let us assume that $a_2(x) \neq 0$ on the interval of interest. We will write our equation in **standard form**

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

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where $P = a_1/a_2$ and $Q = a_0/a_2$.

Generalization

Consider the equation **in standard form** with one known solution. Determine a second linearly independent solution.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \quad y_1(x) - -is \text{ known.}$$

we assume that $y_2(x) = u(x)y_1(x)$ for some function
 $u(x)$. Note, since y_1 , is a solution
 $\frac{d^2y_1}{dx^2} + P(x)\frac{dy_1}{dx} + Q(x)y_1 = 0$

Liell substitute $y_2 = uy_1$ into the ODE.

Well assume for now that $u(x) > 0$.

$$y_{x} = uy_{1}$$

$$y_{z}' = u'y_{1} + uy_{1}'$$

$$y_{z}'' = u''y_{1} + u'y_{1}' + u'y_{1}' + uy_{1}''$$

$$= u''y_{1} + zu'y_{1}' + uy_{1}' + uy_{1}''$$

$$y_{z}'' + Pwyy_{z}' + Qwyy_{z} = 0$$

$$u''y_{1} + zu'y_{1}' + uy_{1}'' + Pwy(u'y_{1} + uy_{1}') + Qwyyy_{1} = 0$$

$$Collect by u_{1}u'_{1}u''$$

$$y_{1}u'' + (zy_{1}' + Pwy_{1})u' + (y_{1}'' + Pwy_{1}' + Qwy_{1})u = 0$$

Note that the highlighted part gives $\frac{d^{2}y_{1}}{dv^{2}} + P(x)\frac{dy_{1}}{dx} + Q(x)y_{1} = 0$ since by solves the original equation. We have the equation for h y, " + (2y' + P(x)y,) " = 0 This is 1st order linear and separable in W. Let w= h'so w'= h", w solves y, w' + (2y, + P(x) y,) w = 0 イロト イ団ト イヨト イヨト 二日

Separating variables

$$y_{,W}' = -(2y'_{,} + P(x_{1}y_{,}))W$$

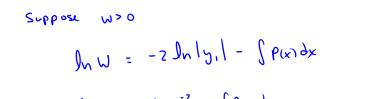
$$\frac{dW}{dx} = -\frac{(2y'_{,} + P(x_{1}y_{,}))}{y_{,}}W$$

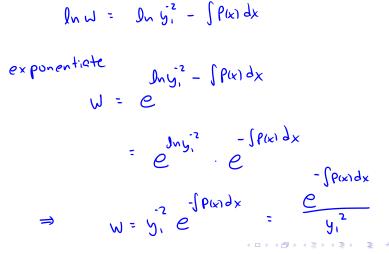
$$\frac{1}{\sqrt{dx}} = -\frac{2y'_{,}}{y_{,}} - P(x)$$

$$\frac{1}{\sqrt{dx}} = -2\frac{dy'_{,}}{y_{,}} - f(x) dx$$

$$\int \frac{1}{\sqrt{dx}} dW = -2\int \frac{dy_{,}}{y_{,}} - \int f(x) dx$$

$$(1 + \sqrt{dx}) = -2\int \frac{dy_{,}}{y_{,}} - \int f(x) dx$$





As
$$w = u'$$
, integrate to get
 $u = \int \frac{-\int P(x) dx}{(y_1(x))^2} dx$
and the 2nd solution

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Reduction of Order Formula

For the second order, homogeneous equation in standard form with one known solution y_1 , a second linearly independent solution y_2 is given by

$$y_2 = y_1(x) \int \frac{e^{-\int P(x) \, dx}}{(y_1(x))^2} \, dx$$

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Example

Solve the initial value problem given one solution to the differential equation.

$$x^{2}y'' - 3xy' + 4y = 0, \quad y_{1} = x^{2}, \quad y(1) = 3, \quad y'(1) = -3$$
Standard forn
$$y'' - \frac{3}{x} \quad y' + \frac{y}{x^{2}} \quad y = 0$$

$$P(x) = -\frac{3}{x} \quad s^{3} \quad -\int P(x) dx = -\int -\frac{3}{x} dx$$

$$= \int \frac{3}{x} dx = 3 \ln |x|$$
for $x > 0$ we can drop the obs. Value bars.
$$r = r = \frac{3}{x} dx = 3 \ln |x|$$
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$$y_2 = uy_1$$
, where $u = \int \frac{-\int P(x) dx}{(y_1)^2} dx$

$$u: \int \frac{\frac{3\ln x}{e}}{(x^{2})^{2}} dx = \int \frac{\ln x^{3}}{x^{3}} dx$$

$$= \int \frac{x^3}{x^m} dx = \int \frac{1}{x} dx = \ln x$$

So
$$y_z = uy_1 = (J_{nx}) X^2 = \chi^2 J_{nx}$$

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The general solution to the ODE is y= C, x2 + C2 x2 hx Applo 5(1)= 3, 5'(1)= -3 y'= 2 (1 × + 2 (2 × Dn× + G. × $y(1) = C_1(1)^2 + C_2(1)^2 dn 1 = 3$ $C_1 = 3$ y'(1)= 20,1+20,1 9/1 + 021 = -3 <ロト < 回 > < 回 > < 三 > < 三 > 三 三 September 19, 2018

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$$\partial (1 + (2 = -3))$$

 $(2 = -3 - 2(1 = -3 - 2.3 = -9)$
The solution to the IVP is
 $y = 3x^{2} - 9x^{2} \ln x$