Sept 21 Math 2306 sec. 53 Fall 2018

Section 7: Reduction of Order

We'll focus on second order, linear, homogeneous equations. Recall that such an equation has the form

$$a_2(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_0(x)y=0.$$

Let us assume that $a_2(x)\neq 0$ on the interval of interest. We will write our equation in **standard form**

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0$$

where $P=a_1/a_2$ and $Q=a_0/a_2$.

Generalization

Consider the equation **in standard form** with one known solution. Determine a second linearly independent solution.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \quad y_1(x) - -\text{is known.}$$

we assume that $y_2(x) = u(x)y_1(x)$ for some function $u(x)$. Note, since y_1 is a solution

$$\frac{d^2y_1}{dx^2} + P(x)\frac{dy_1}{dx} + Q(x)y_1 = 0$$

We'll substitute $y_2 = uy_1$ into the ODE.

We'll assume for now that $u(x) > 0$.

$$y_2 = u y_1$$

$$y_2' = u' y_1 + u y_1'$$

$$y_2'' = u'' y_1 + u' y_1' + u' y_1' + u y_1''$$

$$= u'' y_1 + 2u' y_1' + u y_1''$$

$$y_2'' + P(x) y_2' + Q(x) y_2 = 0$$

$$\underline{u''} y_1 + \underline{2u'} y_1' + \underline{u} y_1'' + P(x)(\underline{u'} y_1 + \underline{u} y_1') + \underline{Q(x) u} y_1 = 0$$

Collect by u, u', u''

$$y_1 u'' + (2y_1' + P(x) y_1) u' + (y_1'' + P(x) y_1' + Q(x) y_1) u = 0$$

Note that the highlighted part gives

$$\frac{d^2 y_1}{dx^2} + P(x)\frac{dy_1}{dx} + Q(x)y_1 = 0$$

since y_1 solves the original equation.

We have the equation for u

$$y_1 u'' + (2y_1' + P(x)y_1)u' = 0$$

This is 1st order linear and separable in u'.

Let $w = u'$ so $w' = u''$, w solves

$$y_1 w' + (2y_1' + P(x)y_1)w = 0$$

Separating variables

$$y_1 w' = -\left(2y_1' + P(x)y_1\right) w$$

$$\frac{dw}{dx} = -\frac{\left(2y_1' + P(x)y_1\right)}{y_1} w$$

$$\frac{1}{w}\frac{dw}{dx} = \frac{-2y_1'}{y_1} - P(x)$$

$$\frac{1}{w}\frac{dw}{dx}\,dx = -2\,\frac{\frac{dy_1}{dx}}{y_1}\,dx - P(x)\,dx$$

$$\int \frac{1}{w}\,dw = -2\int \frac{dy_1}{y_1} - \int P(x)\,dx$$

Suppose $W > 0$

$$\ln W = -2\ln|y_1| - \int P(x)\,dx$$

$$\ln W = \ln y_1^{-2} - \int P(x)\,dx$$

exponentiate

$$W = e^{\ln y_1^{-2} - \int P(x)\,dx}$$

$$= e^{\ln y_1^{-2}} \cdot e^{-\int P(x)\,dx}$$

$$\Rightarrow \quad W = y_1^{-2} e^{-\int P(x)\,dx} = \frac{e^{-\int P(x)\,dx}}{y_1^2}$$

As $w = u'$, integrate to get

$$u = \int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx$$

and the 2nd solution

$$y_2 = u y_1$$

The general solution to the ODE is

$$y = C_1 y_1 + C_2 y_2$$

Reduction of Order Formula

For the second order, homogeneous equation **in standard form** with one known solution y_1, a second linearly independent solution y_2 is given by

$$y_2 = y_1(x)\int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx$$

Example

Solve the initial value problem given one solution to the differential equation.

$$x^2y'' - 3xy' + 4y = 0, \quad y_1 = x^2, \quad y(1) = 3, \quad y'(1) = -3$$

Standard form

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0$$

$P(x) = \frac{-3}{x}$ so $-\int P(x)\,dx = -\int \frac{-3}{x}\,dx$

$$= \int \frac{3}{x}\,dx = 3\ln|x|$$

for $x > 0$ we can drop the abs. value bars.

$y_2 = u y_1$ where $u = \int \frac{e^{-\int P(x)\,dx}}{(y_1)^2}\,dx$

$$u = \int \frac{e^{3\ln x}}{(x^2)^2}\,dx = \int \frac{e^{\ln x^3}}{x^4}\,dx$$

$$= \int \frac{x^3}{x^4}\,dx = \int \frac{1}{x}\,dx = \ln x$$

So $\quad y_2 = u y_1 = (\ln x)\,x^2 = x^2 \ln x$

The general solution to the ODE is

$$y = C_1 x^2 + C_2 x^2 \ln x$$

Apply $\quad y(1) = 3, \quad y'(1) = -3$

$$y' = 2C_1 x + 2C_2 x \ln x + \frac{C_2 x^2}{x}$$

$$y(1) = C_1 (1)^2 + C_2 (1)^2 \ln 1 = 3$$

$$C_1 = 3$$

$$y'(1) = 2C_1 \cdot 1 + 2C_2 \cdot 1 \underbrace{\cancel{\ln 1}}_{0} + C_2 \cdot 1 = -3$$

$$2c_1 + c_2 = -3$$

$$c_2 = -3 - 2c_1 = -3 - 2\cdot 3 = -9$$

The solution to the IVP is

$$y = 3x^2 - 9x^2 \ln x$$