September 21 Math 2306 sec 54 Fall 2015

Section 4.2: Reduction of Order

Recall that a second order, homogeneous, linear equation in standard form looks like
$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0.$$

Any fundamental solution set will necessarily consist of two (y_1, y_2) linearly independent solutions.

Herein, we will assume that one such solution $y_1(x)$ is known. We seek the second in the form

$$y_2(x) = u(x)y_1(x)$$

for some function $u(x)$.

Generalization

Consider the equation **in standard form** with one known solution. Determine a second linearly independent solution.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \quad y_1(x) - -\text{is known.}$$

Look for $y_2(x) = u(x)\, y_1(x)$

$y_2' = u' y_1 + u y_1'$

$y_2'' = u'' y_1 + u' y_1' + u' y_1' + u y_1''$

$= u'' y_1 + 2u' y_1' + u y_1''$

$y_2'' + P(x) y_2' + Q(x) y_2 = 0$

$$u''y_1 + 2u'y_1' + uy_1'' + P(x)\left(u'y_1 + uy_1'\right) + Q(x)uy_1 = 0$$

$$y_1 u'' + \left(2y_1' + P(x)\,y_1\right)u' + \underbrace{\left(y_1'' + P(x)\,y_1' + Q(x)y_1\right)}_{=0}u = 0$$

Note y_1 solves the ODE i.e. $y_1'' + P(x)y_1' + Q(x)y_1 = 0$

$$y_1 u'' + \left(2y_1' + P(x)\,y_1\right)u' = 0$$

Let $w = u'$

so $w' = u''$

$$y_1 w' + \left(2y_1' + P(x)\,y_1\right)w = 0$$

$$w' + \left(\frac{2y_1'}{y_1} + P(x)\right)w = 0$$

$$\frac{dw}{dx} = -\left(\frac{2y_1'}{y_1} + P(x)\right)w$$

$$\frac{1}{w}\frac{dw}{dx}\,dx = -\left(\frac{2y_1'}{y_1} + P(x)\right)dx$$

$$\int \frac{1}{w}\,dw = -\int 2\,\frac{dy_1}{y_1} - \int P(x)\,dx$$

$$\ln w = -2\ln y_1 - \int P(x)\,dx$$

* see 3 slides down for more details of this step

$$\ln w = \ln y_1^{-2} - \int P(x)\,dx$$

$$e^{\ln w} = e^{\ln y_1^{-2} - \int P(x)\,dx} = e^{\ln y_1^{-2}} \cdot e^{-\int P(x)dx}$$

$$\Rightarrow \quad w = y_1^{-2}\, e^{-\int P(x)dx}$$

$$w = \frac{e^{-\int P(x)dx}}{(y_1(x))^2}$$

$w = u'$ so $u = \int w\,dx$

$$u = \int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx$$

and

$$y_2(x) = y_1(x)\,u(x)$$

$*$

$$\frac{1}{w}\frac{dw}{dx}\,dx = -\left(2\,\frac{\frac{dy_1}{dx}}{y_1} + P(x)\right)dx$$

$$\frac{1}{w}\,dw = \frac{-2}{y_1}\frac{dy_1}{dx}\,dx - P(x)\,dx$$

$$\frac{1}{w}\,dw = \frac{-2}{y_1}\,dy_1 - P(x)\,dx$$

Reduction of Order Formula

For the second order, homogeneous equation **in standard form** with one known solution y_1, a second linearly independent solution y_2 is given by

$$y_2 = y_1(x)\int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx$$

Example

Find the general solution of the ODE given one known solution

$$x^2y'' - 3xy' + 4y = 0, \quad y_1 = x^2$$

Lets assume $x>0$ and put the equation in Standard form

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0$$

$$P(x) = -\frac{3}{x}, \quad -\int P(x)\,dx = -\int -\frac{3}{x}\,dx = 3\ln x = \ln x^3$$

So
$$\frac{e^{-\int P(x)\,dx}}{(y_1(x))^2} = \frac{e^{\ln x^3}}{(x^2)^2} = \frac{x^3}{x^4} = \frac{1}{x}$$

$$u(x) = \int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx = \int \frac{1}{x}\,dx = \ln x$$

$$y_2(x) = u(x)\,y_1(x) = x^2 \ln x$$

The general solution to the ODE is

$$y = C_1 y_1(x) + C_2 y_2(x) \implies \boxed{y = C_1 x^2 + C_2 x^2 \ln x}$$

Example

Find the solution of the IVP where one solution of the ODE is given.

$$y'' + 4y' + 4y = 0 \quad y_1 = e^{-2x}, \quad y(0) = 1, \quad y'(0) = \quad 2$$

Use reduction of order to find y_2:

$$P(x) = 4 \, , \quad -\int P(x)\,dx = -\int 4\,dx = -4x$$

$$e^{-\int P(x)\,dx} = e^{-4x}$$

$$u(x) = \int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx = \int \frac{e^{-4x}}{(e^{-2x})^2}\,dx$$

$$= \int \frac{e^{-4x}}{e^{-4x}}\,dx = \int dx = x$$

Hence $y_2(x) = u(x)\,y_1(x) = x e^{-2x}$

The general solution to the DE is

$$y = c_1 e^{-2x} + c_2 x e^{-2x}$$

Apply the I.C. $y(0) = 1$ $y'(0) = 2$

$$y' = -2c_1 e^{-2x} - 2c_2 x e^{-2x} + c_2 e^{-2x}$$

$$y(0) = c_1 e^0 + c_2 \cdot 0\, e^0 = 1 \Rightarrow c_1 = 1$$

$$y'(0) = -2 \cdot 1\, e^0 - 2c_2 \cdot 0\, e^0 + c_2 \cdot e^0 = 2$$

$$-2 + c_2 = 2 \Rightarrow c_2 = 4$$

The solution to the IVP is

$$\boxed{y = e^{-2x} + 4xe^{-2x}.}$$

Section 4.3: Homogeneous Equations with Constant Coefficients

We consider a second order, linear, homogeneous equation with constant coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.$$

Question: What sort of function y could be expected to satisfy

$$y'' = \text{constant} \times y' + \text{constant} \times y?$$

We look for solutions of the form $y = e^{mx}$ with m constant.

This is supposed to solve the DE

$$ay'' + by' + cy = 0$$

$$y = e^{mx}, \quad y' = me^{mx}, \quad y'' = m^2 e^{mx}$$

$$a(m^2 e^{mx}) + b(me^{mx}) + ce^{mx} = 0$$

$$e^{mx}(am^2 + bm + c) = 0$$

This holds for all x in some interval if

$$am^2 + bm + c = 0$$

This is called the characteristic equation for the D.E.

Auxiliary a.k.a. Characteristic Equation

$$am^2 + bm + c = 0$$

There are three cases:

I $b^2 - 4ac > 0$ and there are two distinct real roots $m_1 \neq m_2$

II $b^2 - 4ac = 0$ and there is one repeated real root $m_1 = m_2 = m$

III $b^2 - 4ac < 0$ and there are two roots that are complex conjugates $m_{1,2} = \alpha \pm i\beta$

Case I: Two distinct real roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac > 0$$

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} \quad \text{where} \quad m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Show that $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$ are linearly independent.

$$W(y_1, y_2)(x) = \begin{vmatrix} e^{m_1 x} & e^{m_2 x} \\ m_1 e^{m_1 x} & m_2 e^{m_2 x} \end{vmatrix} = e^{m_1 x}\left(m_2 e^{m_2 x}\right) - m_1 e^{m_1 x}\left(e^{m_2 x}\right)$$

$$= e^{(m_1 + m_2)x}(m_2 - m_1) \neq 0$$

So, y_1 and y_2 are lin. independent.

Since $m_2 \neq m_1$, $m_2 - m_1 \neq 0$