September 21 Math 2306 sec. 56 Fall 2017

Section 7: Reduction of Order

We're considering the equation second order linear homogeneous equation **in standard form**

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0.$$

We are assuming that P and Q are continuous on an interval and that we know one solution $y_1(x)$. Note that this means that

$$\frac{d^2y_1}{dx^2}+P(x)\frac{dy_1}{dx}+Q(x)y_1=0.$$

Any fundamental solution set must contain another linearly independent solution $y_2(x)$.

Method of Reduction of Order: We assume that $y_2(x)=u(x)y_1(x)$ where u is some function that we expect to be able to find.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0 \text{ with } y_1(x) \text{ known}$$

start w/ $y_2 = u y_1$

Find y_2', y_2'' $\quad y_2' = u'y_1 + uy_1'$

$$y_2'' = u''y_1 + u'y_1' + u'y_1' + uy_1''$$

$$= u''y_1 + 2u'y_1' + uy_1''$$

Substitute $\quad \frac{d^2y_2}{dx^2} + P(x)\frac{dy_2}{dx} + Q(x)y_2 = 0$

$$u''y_1 + 2u'y_1' + uy_1'' + P(x)\left(u'y_1 + uy_1'\right) + Q(x)uy_1 = 0$$

Collect like derivatives of u

$$u''y_1 + \left(2y_1' + P(x)\,y_1\right)u' + \underbrace{\left(y_1'' + P(x)\,y_1' + Q(x)\,y_1\right)}_{=0}u = 0$$

Since y_1 solves the homogeneous eqn.

$$y_1 u'' + \left(2y_1' + P(x)y_1\right)u' = 0$$

Let $w = u'$ so $w' = u''$ the eqn is

$$y_1 w' + \left(2y_1' + P(x)y_1\right)w = 0$$

1st order lin and separable

we'll separate variables

$$y_1 \frac{dw}{dx} = -\left(2y_1' + P(x)y_1\right) w$$

we'll assume $w > 0$

$$\frac{1}{w}\frac{dw}{dx} = \frac{-\left(2y_1' + P(x)y_1\right)}{y_1} = \frac{-2y_1'}{y_1} - P(x)$$

$$\int \frac{1}{w}\,dw = \int -2\frac{y_1'}{y_1}\,dx - \int P(x)\,dx$$

$$\ast\ \frac{dy_1}{dx}dx = dy_1$$

$$\int \frac{1}{w}\,dw = -2\int \frac{dy_1}{y_1} - \int P(x)\,dx$$

$$\ln W = -2\ln|y_1| - \int p(x)\,dx$$

$$\ln W = \ln y_1^{-2} - \int p(x)\,dx$$

Exponentiate

$$e^{\ln W} = e^{\ln y_1^{-2} - \int p(x)\,dx}$$

$$W = e^{\ln y_1^{-2}} \cdot e^{-\int p(x)\,dx}$$

$$W = y_1^{-2} e^{-\int p(x)\,dx}$$

$$W = \frac{e^{-\int P(x)\,dx}}{y_1^2}$$

$w = u'$ so $u = \int w\,dx$

so $$u(x) = \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx$$

and $$y_2(x) = u(x)\,y_1(x)$$

Reduction of Order Formula

For the second order, homogeneous equation **in standard form** with one known solution y_1, a second linearly independent solution y_2 is given by

$$y_2 = y_1(x) \int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx$$

Example

Find the general solution of the ODE given one known solution

$$x^2y'' - 3xy' + 4y = 0, \quad y_1 = x^2$$

we'll use reduction of order: Standard form:

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0$$

$P(x) = \frac{-3}{x}$ our integrand is $\dfrac{e^{-\int P(x)dx}}{(y_1)^2}$

The integrand is $\dfrac{e^{-\int \frac{-3}{x}dx}}{(x^2)^2}$

$$\frac{e^{\int \frac{3}{x}\,dx}}{x^4} = \frac{e^{3\ln x}}{x^4} = \frac{e^{\ln x^3}}{x^4} = \frac{x^3}{x^4} = \frac{1}{x}$$

$$u = \int \frac{e^{-\int P(x)\,dx}}{(y_1)^2}\,dx = \int \frac{1}{x}\,dx = \ln x$$

$$y_2 = u y_1 = (\ln x)\, x^2 = x^2 \ln x$$

The general solution to the homogeneous eqn is $y = c_1 x^2 + c_2 x^2 \ln x$

Section 8: Homogeneous Equations with Constant Coefficients

We consider a second order, linear, homogeneous equation with constant coefficients

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0.$$

Question: What sort of function y could be expected to satisfy

$$y''=\text{constant}\,y'+\text{constant}\,y?$$

We look for solutions of the form $y = e^{mx}$ with m constant.

We want to solve $ay'' + by' + cy = 0$

$y = e^{mx}$, $y' = me^{mx}$, $y'' = m^2 e^{mx}$

Substitute $am^2 e^{mx} + bme^{mx} + ce^{mx} = 0$

factor $e^{mx}(am^2 + bm + c) = 0$

This holds for all x in some interval if

$$am^2 + bm + c = 0$$

quadratic equation!

Auxiliary a.k.a. Characteristic Equation

$$am^2 + bm + c = 0$$

There are three cases:

I $b^2 - 4ac > 0$ and there are two distinct real roots $m_1 \neq m_2$

II $b^2 - 4ac = 0$ and there is one repeated real root $m_1 = m_2 = m$

III $b^2 - 4ac < 0$ and there are two roots that are complex conjugates $m_{1,2} = \alpha \pm i\beta$

Case I: Two distinct real roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac > 0$$

$$y = c_1 \underbrace{e^{m_1 x}}_{y_1} + c_2 \underbrace{e^{m_2 x}}_{y_2} \quad \text{where} \quad m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Show that $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$ are linearly independent.

We'll use the Wronskian 2 fnct $\Rightarrow$ 2x2

$$W(y_1, y_2)(x) = \begin{vmatrix} e^{m_1 x} & e^{m_2 x} \\ m_1 e^{m_1 x} & m_2 e^{m_2 x} \end{vmatrix}$$

$$= e^{m_1 x}\left(m_2 e^{m_2 x}\right) - m_1 e^{m_1 x}\left(e^{m_2 x}\right)$$

$$= e^{m_1 x} e^{m_2 x} \left(m_2 - m_1 \right)$$

$$W(y_1, y_2)(x) = e^{(m_1+m_2)x} \left(m_2 - m_1 \right)$$

The claim is that $W \neq 0$.

Well, $e^{(m_1+m_2)x}$ is never zero.

Since $m_1 \neq m_2$, $m_2 - m_1 \neq 0$

So $W \neq 0$, they are lin. independent.

Example

Solve the IVP

$$y'' + y' - 12y = 0, \quad y(0) = 1, \quad y'(0) = 10$$

2nd order linear w/ constant coefficients.

Characteristic Eqn: $m^2 + m - 12 = 0$

$(m+4)(m-3) = 0$ 2 real roots

$m_1 = -4 \quad m_2 = 3$

$y_1 = e^{-4x}, \quad y_2 = e^{3x}$

General solution is $y = c_1 e^{-4x} + c_2 e^{3x}$

Apply $y(0) = 1$ $\quad y'(0) = 10$

$$y = c_1 e^{-4x} + c_2 e^{3x} \qquad y(0) = c_1 + c_2 = 1$$

$$y' = -4c_1 e^{-4x} + 3c_2 e^{3x} \qquad y'(0) = -4c_1 + 3c_2 = 10$$

$$\left.\begin{aligned} 4c_1 + 4c_2 &= 4 \\ -4c_1 + 3c_2 &= 10 \end{aligned}\right\} \overset{\text{add}}{\Longrightarrow} \quad 7c_2 = 14 \quad c_2 = 2$$

$$c_1 = 1 - c_2 = 1 - 2 = -1$$

$$\boxed{\text{The soln to the IVP is} \quad y = -e^{-4x} + 2e^{3x}}$$