Section 7: Reduction of Order

We’re considering the equation second order linear homogeneous equation in standard form

\[ \frac{d^2 y}{dx^2} + P(x) \frac{dy}{dx} + Q(x)y = 0. \]

We are assuming that \( P \) and \( Q \) are continuous on an interval and that we know one solution \( y_1(x) \). Note that this means that

\[ \frac{d^2 y_1}{dx^2} + P(x) \frac{dy_1}{dx} + Q(x)y_1 = 0. \]

Any fundamental solution set must contain another linearly independent solution \( y_2(x) \).

Method of Reduction of Order: We assume that \( y_2(x) = u(x)y_1(x) \) where \( u \) is some function that we expect to be able to find.
\[ \frac{d^2y}{dx^2} + P(x) \frac{dy}{dx} + Q(x)y = 0 \text{ with } y_1(x) \text{ known} \]
Reduction of Order Formula

For the second order, homogeneous equation in standard form with one known solution \( y_1 \), a second linearly independent solution \( y_2 \) is given by

\[
y_2 = y_1(x) \int \frac{e^{-\int P(x) \, dx}}{(y_1(x))^2} \, dx
\]
Example

Find the general solution of the ODE given one known solution

\[ x^2 y'' - 3xy' + 4y = 0, \quad y_1 = x^2 \]
Section 8: Homogeneous Equations with Constant Coefficients

We consider a second order, linear, homogeneous equation with constant coefficients

$$a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = 0.$$ 

Question: What sort of function $y$ could be expected to satisfy

$$y'' = \text{constant} \; y' + \text{constant} \; y?$$
We look for solutions of the form $y = e^{mx}$ with $m$ constant.
Auxiliary a.k.a. Characteristic Equation

\[ am^2 + bm + c = 0 \]

There are three cases:

I  \( b^2 - 4ac > 0 \) and there are two distinct real roots \( m_1 \neq m_2 \)

II  \( b^2 - 4ac = 0 \) and there is one repeated real root \( m_1 = m_2 = m \)

III  \( b^2 - 4ac < 0 \) and there are two roots that are complex conjugates \( m_{1,2} = \alpha \pm i\beta \)
Case I: Two distinct real roots

\[ ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac > 0 \]

\[ y = c_1 e^{m_1 x} + c_2 e^{m_2 x} \quad \text{where} \quad m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

Show that \( y_1 = e^{m_1 x} \) and \( y_2 = e^{m_2 x} \) are linearly independent.
Example

Solve the IVP

\[ y'' + y' - 12y = 0, \quad y(0) = 1, \quad y'(0) = 10 \]
Case II: One repeated real root

\[ ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac = 0 \]

\[ y = c_1 e^{mx} + c_2 x e^{mx} \quad \text{where} \quad m = \frac{-b}{2a} \]

Use reduction of order to show that if \( y_1 = e^{\frac{-bx}{2a}} \), then \( y_2 = xe^{\frac{-bx}{2a}} \).
Example

Solve the ODE

\[ 4y'' - 4y' + y = 0 \]
Case III: Complex conjugate roots

\[ ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac < 0 \]

\[ y = e^{\alpha x} (c_1 \cos(\beta x) + c_2 \sin(\beta x)), \quad \text{where the roots} \]

\[ m = \alpha \pm i\beta, \quad \alpha = \frac{-b}{2a} \quad \text{and} \quad \beta = \frac{\sqrt{4ac - b^2}}{2a} \]

The solutions can be written as

\[ Y_1 = e^{(\alpha + i\beta)x} = e^{\alpha x} e^{i\beta x}, \quad \text{and} \quad Y_2 = e^{(\alpha - i\beta)x} = e^{\alpha x} e^{-i\beta x}. \]
Deriving the solutions Case III

Recall Euler’s Formula:

\[ e^{i\theta} = \cos \theta + i \sin \theta \]
Example

Solve the ODE

\[ \frac{d^2 x}{dt^2} + 4 \frac{dx}{dt} + 6x = 0 \]
Higer Order Linear Constant Coefficient ODEs

- The same approach applies. For an $n^{th}$ order equation, we obtain an $n^{th}$ degree polynomial.
- Complex roots must appear in conjugate pairs (due to real coefficients) giving a pair of solutions $e^{\alpha x} \cos(\beta x)$ and $e^{\alpha x} \sin(\beta x)$.
- If a root $m$ is repeated $k$ times, we get $k$ linearly independent solutions
  
  $$e^{mx}, \ xe^{mx}, \ x^2 e^{mx}, \ldots, \ x^{k-1} e^{mx}$$

or in conjugate pairs cases $2k$ solutions

  $$e^{\alpha x} \cos(\beta x), \ e^{\alpha x} \sin(\beta x), \ xe^{\alpha x} \cos(\beta x), \ xe^{\alpha x} \sin(\beta x), \ldots,$$

  $$x^{k-1} e^{\alpha x} \cos(\beta x), \ x^{k-1} e^{\alpha x} \sin(\beta x)$$

- It may require a computer algebra system to find the roots for a high degree polynomial.
Example

Solve the ODE

\[ y''' - 4y' = 0 \]
Example

Solve the ODE

\[ y''' - 3y'' + 3y' - y = 0 \]