September 21 Math 2306 sec. 57 Fall 2017

Section 7: Reduction of Order

We're considering the equation second order linear homogeneous equation **in standard form**

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0.$$

We are assuming that P and Q are continuous on an interval and that we know one solution $y_1(x)$. Note that this means that

$$\frac{d^2y_1}{dx^2}+P(x)\frac{dy_1}{dx}+Q(x)y_1=0.$$

Any fundamental solution set must contain another linearly independent solution $y_2(x)$.

Method of Reduction of Order: We assume that $y_2(x)=u(x)y_1(x)$ where u is some function that we expect to be able to find.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0 \text{ with } y_1(x) \text{ known}$$

we look for y_2 in the form $y_2 = uy_1$

we sub into the ODE

$$y_2 = uy_1$$

$$y_2' = u'y_1 + uy_1'$$

$$y_2'' = u''y_1 + u'y_1' + u'y_1' + uy_1''$$

$$= u''y_1 + 2u'y_1' + uy_1''$$

Substitute

$$\frac{d^2y_2}{dx^2} + P(x)\frac{dy_2}{dx} + Q(x)y_2 = 0$$

$$u''y_1 + 2u'y_1' + uy_1'' + P(x)(u'y_1 + uy_1') + Q(x)uy_1 = 0$$

Let's collect by derivatives of u

$$y_1u'' + (2y_1' + P(x)y_1)u' + (\underbrace{y_1'' + P(x)y_1' + Q(x)y_1}_{=0})u = 0$$

since y_1 solves the homogeneous eqn.

$$y_1u'' + (2y_1' + P(x)y_1)u' = 0$$

Let $w = u'$ so $w' = u''$, then w solves

$$y_1 w' + (2y_1' + P(x) y_1) w = 0$$

1st order linear and separable

Let's separate the variables

$$y_1 \frac{dw}{dx} = -(2y_1' + P(x)y_1) w$$

We'll assume $w > 0$.

$$\frac{1}{w} \frac{dw}{dx} = \frac{-(2y_1' + P(x)y_1)}{y_1} = -\frac{2y_1'}{y_1} - P(x)$$

$$\int \frac{1}{w}\, dw = \int \frac{-2y_1'}{y_1} dx - \int P(x)\, dx$$

Note $\quad y_1' \, dx = \frac{dy_1}{dx} dx = dy_1$

$$\int \frac{1}{w} \, dw = \int -2 \frac{dy_1}{y_1} - \int P(x) \, dx$$

$$\ln w = -2 \ln |y_1| - \int P(x) \, dx$$

$$\ln w = \ln y_1^{-2} - \int P(x) \, dx$$

$$e^{\ln w} = e^{\ln y_1^{-2} - \int P(x) dx}$$

$$= e^{\ln y_1^{-2}} \cdot e^{-\int P(x) dx}$$

$$w = y_1^{-2} e^{-\int P(x)\,dx} = \frac{e^{-\int P(x)\,dx}}{y_1^2}$$

$$w = u' \quad \text{so} \quad u = \int w\,dx$$

$$u = \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx$$

Finally $$y_2 = u(x)\,y_1(x)$$

Reduction of Order Formula

For the second order, homogeneous equation **in standard form** with one known solution y_1, a second linearly independent solution y_2 is given by

$$y_2 = y_1(x) \int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx$$

Example

Find the general solution of the ODE given one known solution

$$x^2y'' - 3xy' + 4y = 0, \quad y_1 = x^2$$

We'll assume $x > 0$. In standard form

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0$$

$P(x) = \frac{-3}{x}$ our integrand is $\dfrac{e^{-\int P(x)\,dx}}{y_1^2}$

This is $\dfrac{e^{-\int \frac{-3}{x}\,dx}}{(x^2)^2}$

$$\frac{e^{\int \frac{3}{x}\,dx}}{x^4} = \frac{e^{3\ln x}}{x^4} = \frac{e^{\ln x^3}}{x^4} = \frac{x^3}{x^4} = \frac{1}{x}$$

$$u = \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx = \int \frac{1}{x}\,dx = \ln x$$

$$y_2 = u y_1 = (\ln x)x^2 = x^2 \ln x$$

The general solution is

$$\boxed{y = c_1 x^2 + c_2 x^2 \ln x}$$

Section 8: Homogeneous Equations with Constant Coefficients

We consider a second order, linear, homogeneous equation with constant coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.$$

Question: What sort of function y could be expected to satisfy

$$y'' = \text{constant}\, y' + \text{constant}\, y?$$

We look for solutions of the form $y = e^{mx}$ with m constant.

We'll sub this into the ODE $ay'' + by' + cy = 0$

$y = e^{mx}$, $y' = me^{mx}$, $y'' = m^2 e^{mx}$

$$a(m^2 e^{mx}) + bme^{mx} + ce^{mx} = 0$$

factor $$e^{mx}(am^2 + bm + c) = 0$$

This holds for all x in some interval if

$$am^2 + bm + c = 0$$ quadratic Eqn!

Auxiliary a.k.a. Characteristic Equation

$$am^2 + bm + c = 0$$

There are three cases:

I $b^2 - 4ac > 0$ and there are two distinct real roots $m_1 \neq m_2$

II $b^2 - 4ac = 0$ and there is one repeated real root $m_1 = m_2 = m$

III $b^2 - 4ac < 0$ and there are two roots that are complex conjugates $m_{1,2} = \alpha \pm i\beta$

Case I: Two distinct real roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac > 0$$

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} \quad \text{where} \quad m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Show that $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$ are linearly independent.

We'll use the Wronskian. 2 fnct $\Rightarrow$ 2×2

$m_1 \neq m_2$

$$W(y_1, y_2)(x) = \begin{vmatrix} e^{m_1 x} & e^{m_2 x} \\ m_1 e^{m_1 x} & m_2 e^{m_2 x} \end{vmatrix}$$

$$= e^{m_1 x}\left(m_2 e^{m_2 x}\right) - m_1 e^{m_1 x}\left(e^{m_2 x}\right)$$

$$= e^{m_1 x} e^{m_2 x} (m_2 - m_1)$$

The claim is that $W \neq 0$.

Note $e^{m_1 x} e^{m_2 x} > 0$

$m_2 - m_1 \neq 0$ because $m_1 \neq m_2$

$$W(y_1, y_2)(x) = e^{(m_1+m_2)x} (m_2 - m_1) \neq 0$$

So y_1, y_2 are linearly independent.

Example

Solve the IVP

$$y'' + y' - 12y = 0, \quad y(0) = 1, \quad y'(0) = 10$$

2nd order linear constant coeff.

Characterist eqn: $m^2 + m - 12 = 0$ 2 real roots

$$(m+4)(m-3) = 0$$

$$m_1 = -4, \quad m_2 = 3$$

$y_1 = e^{-4x}$, $y_2 = e^{3x}$ the general soln. is

$$y = c_1 e^{-4x} + c_2 e^{3x}$$

Impose the IC $y(0)=1$, $y'(0)=10$

$$y = c_1 e^{-4x} + c_2 e^{3x} \qquad y(0) = c_1 + c_2 = 1$$

$$y' = -4c_1 e^{-4x} + 3c_2 e^{3x} \qquad y'(0) = -4c_1 + 3c_2 = 10$$

$$\left.\begin{array}{l} 4c_1 + 4c_2 = 4 \\ -4c_1 + 3c_2 = 10 \end{array}\right\} \overset{\text{add}}{\Longrightarrow} \quad 7c_2 = 14 \Rightarrow c_2 = 2$$

$$c_1 = 1 - c_2 = 1 - 2 = -1$$

The solution to the IVP is

$$\boxed{y = -e^{-4x} + 2e^{3x}.}$$