Section 2.3: Characterization of Invertible Matrices

Given an $n \times n$ matrix $A$, we can think of

- A matrix equation $Ax = b$;
- A linear system that has $A$ as its coefficient matrix;
- A linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by $T(x) = Ax$;
- Not to mention things like its pivots, its rref, the linear dependence/independence of its columns, blah blah blah blah...

**Question:** How is this stuff related, and how does being singular or invertible tie in?
Theorem: Suppose $A$ is $n \times n$. The following are equivalent. ¹

(a) $A$ is invertible.
(b) $A$ is row equivalent to $I_n$.
(c) $A$ has $n$ pivot positions.
(d) $Ax = 0$ has only the trivial solution.
(e) The columns of $A$ are linearly independent.
(f) The transformation $x \mapsto Ax$ is one to one.
(g) $Ax = b$ is consistent for every $b$ in $\mathbb{R}^n$.
(h) The columns of $A$ span $\mathbb{R}^n$.
(i) The transformation $x \mapsto Ax$ is onto.
(j) There exists an $n \times n$ matrix $C$ such that $CA = I$.
(k) There exists an $n \times n$ matrix $D$ such that $AD = I$.
(l) $A^T$ is invertible.

¹Meaning all are true or none are true.
Theorem: (An inverse matrix is unique.)

Let $A$ and $B$ be $n \times n$ matrices. If $AB = I$, then $A$ and $B$ are both invertible with $A^{-1} = B$ and $B^{-1} = A$.

Suppose $B\bar{x} = \bar{0}$. Then

$$\bar{x} = I\bar{x} = AB\bar{x} = A(B\bar{x}) = A\bar{0} = \bar{0}.$$  Thus $B\bar{x} = \bar{0}$ has only the trivial solution.

Hence $B$ has an inverse $B^{-1}$.

Now $AB = I$ mult. on the right by $B^{-1}$

$$AB \bar{B}^{-1} = I \bar{B}^{-1}$$
\[ A (B B'^{-1}) = B \]

\[ A I = B' = A B' \]

Since \( B \) being invertible implies \( B' \)
is invertible

\[ (A) = (B'^{-1})' = B \]

i.e. \( B = A' \).
Invertible Linear Transformations

**Definition:** A linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is said to be **invertible** if there exists a function $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that both

$$S(T(x)) = x \quad \text{and} \quad T(S(x)) = x$$

for every $x$ in $\mathbb{R}^n$.

If such a function exists, we typically denote it by

$$S = T^{-1}.$$
Theorem (Invertibility of a linear transformation and its matrix)

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear transformation and $A$ its standard matrix. Then $T$ is invertible if and only if $A$ is invertible. Moreover, if $T$ is invertible, then

$$T^{-1}(x) = A^{-1}x$$

for every $x$ in $\mathbb{R}^n$. 
Example

Use the standard matrix to determine if the linear transformation is invertible. If it is invertible, characterize the inverse transformation.

\[ T : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \text{ given by } T(x_1, x_2) = (3x_1 - x_2, 4x_2). \]

\[ T(\vec{e}_1) = T(1, 0) = (3, 0), \quad T(\vec{e}_2) = T(0, 1) = (-1, 4). \]

\[ A = \begin{bmatrix} 3 & -1 \\ 0 & 4 \end{bmatrix} \quad \text{det}(A) = 3 \cdot 4 - 0 \cdot (-1) = 12 \neq 0 \]

\[ A \text{ is invertible hence } T \text{ is invertible.} \]

\[ A^{-1} = \frac{1}{12} \begin{bmatrix} 4 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{12} \\ 0 & \frac{1}{4} \end{bmatrix} \]
\[ T^{-1}(\hat{x}) = \hat{A}^{-1} \hat{x} \quad \hat{A}^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} x_1 + \frac{1}{12} x_2 \\ \frac{1}{4} x_2 \end{bmatrix} \]

\[ T^{-1}(\hat{x}) = \left( \frac{1}{3} x_1 + \frac{1}{12} x_2, \frac{1}{4} x_2 \right) \]
Example

Suppose $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a one to one linear transformation. Can we determine whether $T$ is onto? Why (or why not)?

Yes, $T$ is onto.

For linear transformations with square standard matrices, being one to one implies also being onto.
Section 3.1: Introduction to Determinants

Recall that a $2 \times 2$ matrix is invertible if and only if the number called its determinant is nonzero. We had

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{21}a_{12}.$$ 

We wish to extend the concept of determinant to $n \times n$ matrices in general. And we wish to do so in such a way that invertibility holds if and only if the determinant is nonzero.
Determinant $3 \times 3$ case:

Suppose we start with a $3 \times 3$ invertible matrix. And suppose that $a_{11} \neq 0$. We can multiply the second and third rows by $a_{11}$ and begin row reduction.

$$
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix}
\sim
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{11}a_{21} & a_{11}a_{22} & a_{11}a_{23} \\
a_{11}a_{31} & a_{11}a_{32} & a_{11}a_{33}
\end{bmatrix}
\sim
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\
0 & a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{13}a_{31}
\end{bmatrix}
$$
Determinant $3 \times 3$ case continued...

$$
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\
  0 & a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{13}a_{31}
\end{bmatrix}
$$

If $A \sim I$, one of the entries in the 2,2 or the 3,2 position must be nonzero. Let’s assume it is the 2,2 entry. Continue row reduction to get

$$
A \sim 
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\
  0 & 0 & a_{11} \Delta
\end{bmatrix}.
$$

Again, if $A$ is invertible, it must be that the bottom right entry is nonzero. That is

$$
\Delta \neq 0.
$$

Note that if $\Delta = 0$, the rref of $A$ is not $I$—$A$ would be singular.
Determinant 3 × 3 case continued...

With a little rearrangement, we have

\[
\Delta = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})
\]

\[
= a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}
\]

The number \( \Delta \) will be called the \textbf{determinant} of \( A \).
Definitions: Minors

Let \( n \geq 2 \). For an \( n \times n \) matrix \( A \), let \( A_{ij} \) denote the \((n - 1) \times (n - 1)\) matrix obtained from \( A \) by deleting the \( i^{th} \) row and the \( j^{th} \) column of \( A \).

For example, if

\[
A = \begin{bmatrix}
-1 & 3 & 2 & 0 \\
4 & 4 & 0 & -3 \\
-2 & 1 & 7 & 2 \\
3 & 0 & -1 & 6
\end{bmatrix}
\]

then \( A_{23} = \begin{bmatrix}
-1 & 3 & 0 \\
-2 & 1 & 2 \\
3 & 0 & 6
\end{bmatrix} \).

Definition: The \( i, j^{th} \) minor of the \( n \times n \) matrix \( A \) is the number

\[
M_{ij} = \det(A_{ij}).
\]
Definitions: Cofactor

**Definition:** Let $A$ be an $n \times n$ matrix with $n \geq 2$. The $i, j^{th}$ cofactor of $A$ is the number

$$C_{ij} = (-1)^{i+j} M_{ij}.$$

**Example:** Find the three minors $M_{11}, M_{12}, M_{13}$ and find the 3 cofactors $C_{11}, C_{12}, C_{13}$ of the matrix

$$A = \begin{bmatrix}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    a_{31} & a_{32} & a_{33}
\end{bmatrix}$$