September 21 Math 3260 sec. 58 Fall 2017

Section 2.2: Inverse of a Matrix

If A is an $n \times n$ matrix, we seek a matrix A^{-1} that satisfies the condition

$$A^{-1}A = AA^{-1} = I_n.$$

If such matrix A^{-1} exists, we'll say that A is **nonsingular** (a.k.a. *invertible*). Otherwise, we'll say that A is **singular**.

Theorem (2×2 case)

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then A is singular.

The quantity $ad - bc$ is called the **determinant** of A.

Theorem: If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Example

Solve the system using a matrix inverse

$$\begin{array}{rcrcl} 4x_1 & + & x_2 & = & 7 \\ -2x_1 & + & 3x_2 & = & 7 \end{array}$$

$A\vec{x} = \vec{b}$ $\quad A = \begin{bmatrix} 4 & 1 \\ -2 & 3 \end{bmatrix}$ $\quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\quad \vec{b} = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$

$\det(A) = 4 \cdot 3 - (-2)(1) = 14 \neq 0$

A^{-1} exists and

$$A^{-1} = \frac{1}{14}\begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{3}{14} & \frac{-1}{14} \\ \frac{2}{14} & \frac{4}{14} \end{bmatrix}$$

The solution

$$\vec{x} = A^{-1}\vec{b} = \frac{1}{14}\begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

$$= \frac{1}{14}\begin{bmatrix} 14 \\ 42 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Theorem

(i) If A is invertible, then A^{-1} is also invertible and

$$\left(A^{-1}\right)^{-1} = A.$$

(ii) If A and B are invertible $n \times n$ matrices, then the product AB is also invertible[1] with

$$(AB)^{-1} = B^{-1}A^{-1}.$$

(iii) If A is invertible, then so is A^T. Moreover

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T.$$

[1] This can generalize to the product of k invertible matrices.

Elementary Matrices

Definition: An **elementary** matrix is a square matrix obtained from the identity by performing one elementary row operation.

Examples:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$3R_2 \to R_2$ $\qquad$ $2R_1 + R_3 \to R_3$ $\qquad$ $R_1 \leftrightarrow R_2$

Action of Elementary Matrices

Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, and compute the following products

$$E_1 A, \quad E_2 A, \quad \text{and} \quad E_3 A.$$

$$E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ 3d & 3e & 3f \\ g & h & i \end{bmatrix}$$

$3R_2 \to R_2$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_2 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$= \begin{bmatrix} a & b & c \\ d & e & f \\ 2a+g & 2b+h & 2c+i \end{bmatrix}$$

$$2R_1 + R_3 \to R_3$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$E_3 A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$= \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}$$

$$R_1 \leftrightarrow R_2$$

$$E_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Remarks

- Elementary row operations can be equated with matrix multiplication (multiply on the left by an elementary matrix),
- Each elementary matrix is invertible where the inverse *undoes* the row operation,
- Reduction to rref is a sequence of row operations, so it is a sequence of matrix multiplications

$$\mathrm{rref}(A) = E_k \cdots E_2 E_1 A.$$

Theorem

An $n \times n$ matrix A is invertible if and only if it is row equivalent to the identity matrix I_n. Moreover, if

$$\mathrm{rref}(A) = E_k \cdots E_2 E_1 A = I_n, \quad \text{then} \quad A = (E_k \cdots E_2 E_1)^{-1} I_n.$$

That is,

$$A^{-1} = \left[(E_k \cdots E_2 E_1)^{-1} \right]^{-1} = E_k \cdots E_2 E_1.$$

The sequence of operations that reduces A to I_n, transforms I_n into A^{-1}.

This last observation—operations that take A to I_n also take I_n to A^{-1}—gives us a method for computing an inverse!

Algorithm for finding A^{-1}

To find the inverse of a given matrix A:

- Form the $n \times 2n$ augmented matrix $[A \quad I]$.
- Perform whatever row operations are needed to get the first n columns (the A part) to rref.
- If rref(A) is I, then $[A \quad I]$ is row equivalent to $[I \quad A^{-1}]$, and the inverse A^{-1} will be the last n columns of the reduced matrix.
- If rref(A) is NOT I, then A is not invertible.

Remarks: We don't need to know ahead of time if A is invertible to use this algorithm.
If A is singular, we can stop as soon as it's clear that rref$(A) \neq I$.

Examples: Find the Inverse if Possible

(a) $\begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 3 \\ -2 & -6 & 4 \end{bmatrix}$

Set up $[A \quad I]$

$$\left[\begin{array}{cccccc} 1 & 2 & -1 & 1 & 0 & 0 \\ -4 & -7 & 3 & 0 & 1 & 0 \\ -2 & -6 & 4 & 0 & 0 & 1 \end{array}\right]$$

$4R_1 + R_2 \to R_2$

$2R_1 + R_3 \to R_3$

$$\begin{array}{cccccc} 4 & 8 & -4 & 4 & 0 & 0 \\ -4 & -7 & 3 & 0 & 1 & 0 \end{array}$$

$$\begin{array}{cccccc} 2 & 4 & -2 & 2 & 0 & 0 \\ -2 & -6 & 4 & 0 & 0 & 1 \end{array}$$

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 4 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{bmatrix}$$

$2R_2 + R_3 \to R_3$

$$\begin{matrix} 0 & 2 & -2 & 8 & 2 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 10 & 2 & 1 \end{bmatrix}$$

A has only 2 pivot columns so

$\text{rref}(A) \neq I$

A is singular.

Examples: Find the Inverse if Possible

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix} = A$

Form $[A \; I]$

$$\left[\begin{array}{cccccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 5 & 6 & 0 & 0 & 0 & 1 \end{array}\right]$$

$-5R_1 + R_3 \to R_3$

$$\begin{array}{cccccc} -5 & -10 & -15 & -5 & 0 & 0 \\ 5 & 6 & 0 & 0 & 0 & 1 \end{array}$$

$$\left[\begin{array}{cccccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & -4 & -15 & -5 & 0 & 1 \end{array}\right]$$

$4R_2 + R_3 \to R_3$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{bmatrix}$$

$$\begin{matrix} 0 & 4 & 16 & 0 & 4 & 0 \\ 0 & -4 & -15 & -5 & 0 & 1 \end{matrix}$$

$-4R_3 + R_2 \to R_2$

$-3R_3 + R_1 \to R_1$

$$\begin{bmatrix} 1 & 2 & 0 & 16 & -12 & -3 \\ 0 & 1 & 0 & 20 & -15 & -4 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{bmatrix}$$

$$\begin{matrix} 0 & 0 & -4 & 20 & -16 & -4 \\ 0 & 1 & 4 & 0 & 1 & 0 \end{matrix}$$

$$\begin{matrix} 0 & 0 & -3 & 15 & -12 & -3 \\ 1 & 2 & 3 & 1 & 0 & 0 \end{matrix}$$

Finally $-2R_2 + R_1 \to R_1$

$$\begin{bmatrix} 1 & 0 & 0 & -24 & 18 & 5 \\ 0 & 1 & 0 & 20 & -15 & -4 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{bmatrix}$$

$0 \quad -2 \quad 0 \quad -40 \quad 30 \quad 8$

$1 \quad 2 \quad 0 \quad 16 \quad -12 \quad -3$

$$A^{-1} = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}$$

Solve the linear system if possible

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 3 \\ & & x_2 & + & 4x_3 & = & 3 \\ 5x_1 & + & 6x_2 & & & = & 4 \end{array}$$

The coefficient matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$

has inverse $A^{-1} = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}$

$$\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$$

$-24 \cdot 3 + 18 \cdot 3 + 5 \cdot 4$

$20 \cdot 3 + (-15) \cdot 3 + (-4) \cdot 4$

$-5 \cdot 3 + 4 \cdot 3 + 1 \cdot 4$

$$= \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Section 2.3: Characterization of Invertible Matrices

Given an $n \times n$ matrix A, we can think of

- A matrix equation $A\mathbf{x} = \mathbf{b}$;
- A linear system that has A as its coefficient matrix;
- A linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by $T(\mathbf{x}) = A\mathbf{x}$;
- Not to mention things like its **pivots**, its **rref**, the linear dependence/independence of its columns, blah blah blah...

Question: How is this stuff related, and how does being singular or invertible tie in?

Theorem: Suppose A is $n \times n$. The following are equivalent. [2]

(a) A is invertible.
(b) A is row equivalent to I_n.
(c) A has n pivot positions.
(d) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
(e) The columns of A are linearly independent.
(f) The transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one to one.
(g) $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbb{R}^n$.
(h) The columns of A span $\mathbb{R}^n$.
(i) The transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto.
(j) There exists an $n \times n$ matrix C such that $CA = I$.
(k) There exists an $n \times n$ matrix D such that $AD = I$.
(l) A^T is invertible.

[2] Meaning all are true or none are true.

Theorem: (An inverse matrix is unique.)

Let A and B be $n \times n$ matrices. If $AB = I$, then A and B are both invertible with $A^{-1} = B$ and $B^{-1} = A$.

Consider the homogeneous equation $B\vec{x} = \vec{0}$.

$$\vec{x} = I\vec{x} = AB\vec{x} = A(B\vec{x}) = A\vec{0} = \vec{0}.$$

So the homogeneous equation has only the trivial solution so B^{-1} exists.

Now $AB = I$ multiply on the right by B^{-1}

$$ABB^{-1} = IB^{-1}$$

$$A(BB^{-1}) = B^{-1}$$

$$AI = B^{-1} \Rightarrow A = B^{-1}.$$

Since B^{-1} is invertible

$$(A)^{-1} = (B^{-1})^{-1} \Rightarrow A^{-1} = B.$$

Invertible Linear Transformations

Definition: A linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is said to be **invertible** if there exists a function $S : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that both

$$S(T(\mathbf{x})) = \mathbf{x} \quad \text{and} \quad T(S(\mathbf{x})) = \mathbf{x}$$

for every $\mathbf{x}$ in $\mathbb{R}^n$.

If such a function exists, we typically denote it by

$$S = T^{-1}.$$

Theorem (Invertibility of a linear transformation and its matrix)

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear transformation and A its standard matrix. Then T is invertible if and only if A is invertible. Moreover, if T is invertible, then

$$T^{-1}(\mathbf{x}) = A^{-1}\mathbf{x}$$

for every $\mathbf{x}$ in $\mathbb{R}^n$.

Example

Use the standard matrix to determine if the linear transformation is invertible. If it is invertible, characterize the inverse transformation.

$$T:\mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad \text{given by} \quad T(x_1,x_2) = (3x_1 - x_2, 4x_2).$$

Find the standard matrix A.

$T(\vec{e}_1) = T(1,0) = (3,0)$, $T(\vec{e}_2) = T(0,1) = (-1,4)$

$A = \begin{bmatrix} 3 & -1 \\ 0 & 4 \end{bmatrix}$ $\det(A) = 3\cdot 4 - 0\cdot(-1) = 12 \neq 0$

A is invertible

$$A^{-1} = \frac{1}{12}\begin{bmatrix} 4 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{12} \\ 0 & \frac{1}{4} \end{bmatrix}$$

T^{-1} exists and $T^{-1}(\vec{x}) = A^{-1}\vec{x}$

$$A^{-1}\vec{x} = \begin{bmatrix} \frac{1}{3} & \frac{1}{12} \\ 0 & \frac{1}{4} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}x_1 + \frac{1}{12}x_2 \\ \frac{1}{4}x_2 \end{bmatrix}$$

$$T^{-1}(x_1, x_2) = \left(\frac{1}{3}x_1 + \frac{1}{12}x_2, \ \frac{1}{4}x_2 \right)$$

Example

Suppose $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a one to one linear transformation. Can we determine whether T is onto? Why (or why not)?

Yes, we can say T is onto. Its standard matrix is $n \times n$. If $x \mapsto A\vec{x}$ is one to one, it must be onto.