# Sept. 23 Math 1190 sec. 51 Fall 2016

## Section 3.1: The Chain Rule

**Theorem: (Chain Rule)** Suppose $g$ is differentiable at $x$ and $f$ is differentiable at $g(x)$. Then the composite function

$$F = f \circ g$$

is differentiable at $x$ and

$$\frac{d}{dx}F(x) = \frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

In Liebniz notation: if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

# Example

Evaluate $\dfrac{d}{ds}\csc(e^s)$

$$\frac{d}{ds}\csc(e^s) =$$

$$-\csc(e^s)\cot(e^s)\cdot e^s$$

$$= -e^s\csc(e^s)\cot(e^s)$$

Inside function

$$g(s) = e^s$$

$$g'(s) = e^s$$

Outside function

$$f(u) = \csc u$$

$$f'(u) = -\csc u \cot u$$

To evaluate $\csc(e^s)$

(1) compute $e^s$

(2) then take the cosecant

# Example

Find the equation of the line tangent to the graph of $y = \cos^2 x$ at the point $(\frac{\pi}{4}, \frac{1}{2})$.

We need a point and a slope.

point : $(\pi/4, \frac{1}{2})$ given

slope : $m_{tan} = y'(c)$ here $c = \pi/4$

the x-value of the given point.

Find $\frac{dy}{dx}$

$y = (\cos x)^2$

To evaluate

(1) take the cosine

(2) square the result

Chain rule

$\frac{d}{dx} f(g(x)) = f'(g(x))\, g'(x)$

$u = g(x) = \cos x \;,\; f(u) = u^2$

$g'(x) = -\sin x \;,\; f'(u) = 2u$

$$\frac{dy}{dx} = \overset{f'(u)}{2u}\,\overset{g'(x)}{(-\sin x)} = -2\overset{u}{\cos x}\,\sin x$$

$$m_{tan} = y'\left(\frac{\pi}{4}\right) = -2\cos\frac{\pi}{4}\,\sin\frac{\pi}{4} = -2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{-2}{2} = -1$$

we have slope $\overset{m}{-1}$ and point $\left(\overset{x_0}{\pi/4}, \overset{y_0}{\frac{1}{2}}\right)$

$$y - y_0 = m(x - x_0)$$

$$y - \frac{1}{2} = -1\left(x - \pi/4\right)$$

$$\boxed{y = -x + \frac{\pi}{4} + \frac{1}{2}}$$

* We get the slope from the derivative $y'(x)$
* Slope is a number (constant) $y'(c)$.

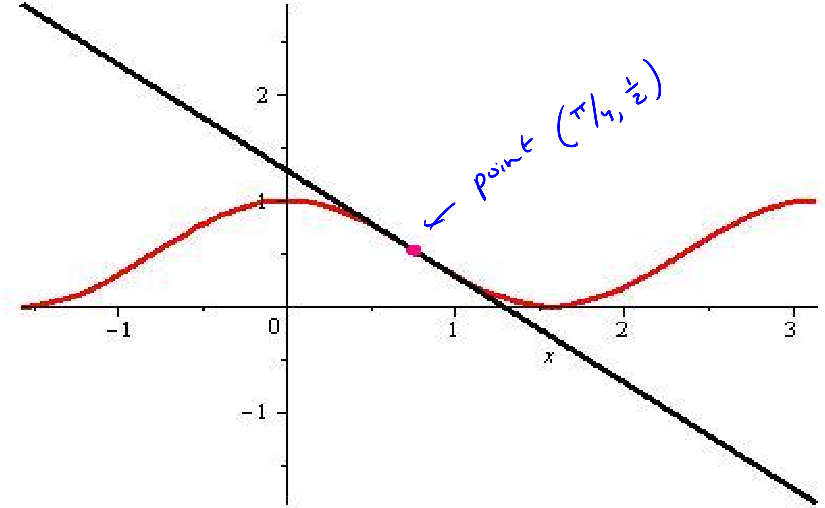


Figure: $y = \cos^2 x$ and the tangent line at $(\frac{\pi}{4}, \frac{1}{2})$.

We may be able to choose between differentiation methods.

Evaluate $\dfrac{d}{dx}\ \dfrac{\sin x}{x^3+2}$ using

(a) The quotient rule:

Quotient Rule

$$\frac{d}{dx}\frac{f}{g} = \frac{f'g - fg'}{g^2}$$

$$\frac{d}{dx}\frac{\sin x}{x^3+2} = \frac{\cos x\,(x^3+2) - \sin x\,(3x^2)}{(x^3+2)^2}$$

$$= \frac{(x^3+2)\cos x - 3x^2\sin x}{(x^3+2)^2}$$

(b) writing $\dfrac{\sin x}{x^3+2} = (\sin x)(x^3+2)^{-1}$ and using the chain rule.

Note $\dfrac{d}{dx}(x^3+2)^{-1}$

$= -u^{-2}(3x^2)$

$= -(x^3+2)^{-2}(3x^2)$

$u = g(x) = x^3+2$ inside

$f(u) = u^{-1}$ outside

$g'(x) = 3x^2$

$f'(u) = -1\,u^{-2} = \dfrac{-1}{u^2}$

$\dfrac{d}{dx}\sin x\,(x^3+2)^{-1} = \cos x\,(x^3+2)^{-1} + \sin x\left(-3x^2(x^3+2)^{-2}\right)$

$= \dfrac{\cos x}{x^3+2} - \dfrac{3x^2 \sin x}{(x^3+2)^2}$

Product rule

$\dfrac{d}{dx}fg = f'g + fg'$

$$= \frac{\cos x}{x^3+2} \cdot \left(\frac{x^3+2}{x^3+2}\right) - \frac{3x^2 \sin x}{(x^3+2)^2}$$

$$= \frac{(x^3+2)\cos x - 3x^2 \sin x}{(x^3+2)^2}$$

which is what we obtained with the quotient rule.

# Questions

Evaluate $f'(x)$ where $f(x) = \sin(3x^2+2x)$

$u = g(x) = 3x^2+2x$

$f(u) = \sin u$

$f'(x) = \cos(3x^2+2x)\cdot(6x+2)$

(a) $f'(x) = \cos(3x^2+2x)$

(b) $f'(x) = (6x+2)\sin(x)+(3x^2+2x)\cos(x)$

(c) $f'(x) = -(6x+2)\cos(3x^2+2x)$

(d) $f'(x) = (6x+2)\cos(3x^2+2x)$

# Questions

Find the equation of the line tangent to the graph of $f(x) = e^{\sin x}$ at the point $(0, f(0))$.

$f(0) = e^{\sin 0} = e^{0} = 1 \qquad \text{pt } (0,1)$

(a) $y = x + 1$

$f'(x) = e^{\sin x} \cdot \cos x$

(b) $y = 1$

$m_{tan} = f'(0) = e^{\sin 0} \cdot \cos 0$

$= e^{0} \cdot 1 = 1 \cdot 1 = 1$

(c) $y = x - 1$

(d) $y = ex + 1$

# Multiple Compositions

The chain rule can be iterated to account for multiple compositions. For example, suppose $f$, $g$, $h$ are appropriately differentiable, then

$$\frac{d}{dx}(f \circ g \circ h)(x) = \frac{d}{dx} f(g(h(x)) = f'(g(h(x))\, g'(h(x))\, h'(x)$$

Note that the outermost function is $f$, and its inner function is a composition $g(h(x))$.

So the derivative of the outer function evaluated at the inner is $f'(g(h(x))$ which is multiplied by the derivative of the inner function—**itself based on the chain rule**—$g'(h(x))h'(x)$.

If $h(x) = v$, $g(v) = u$, and $y = f(u)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}$$

# Example

Evaluate the derivative $\dfrac{d}{dt}\tan^2\left(\dfrac{1}{3}t^3\right) = \dfrac{d}{dt}\left(\tan\left(\frac{1}{3}t^3\right)\right)^2$

$v = h(t) = \frac{1}{3}t^3$ , $u = g(v) = \tan v$ , $f(u) = u^2$

$h'(t) = \frac{1}{3}\cdot 3t^2$ , $g'(v) = \sec^2 v$ , $f'(u) = 2u$

$= t^2$

$$\frac{d}{dt}\tan^2\left(\frac{1}{3}t^3\right) = 2u \cdot \sec^2 v \cdot t^2$$

$$= 2u \cdot \sec^2\left(\frac{1}{3}t^3\right)\cdot t^2$$

$$= 2 \tan v \sec^2\left(\tfrac{1}{3}t^3\right) t^2$$

$$= \underbrace{2 \tan\left(\tfrac{1}{3}t^3\right)}_{f'(g(h(t)))} \underbrace{\sec^2\left(\tfrac{1}{3}t^3\right)}_{g'(h(t))} \cdot \underbrace{t^2}_{h'(t)}$$