Sept. 23 Math 1190 sec. 52 Fall 2016

Section 3.1: The Chain Rule

Theorem: (Chain Rule) Suppose g is differentiable at x and f is differentiable at $g(x)$. Then the composite function

$$F = f \circ g$$

is differentiable at x and

$$\frac{d}{dx}F(x) = \frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

In Liebniz notation: if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Example

Evaluate $\dfrac{d}{ds}\csc(e^s)$

Inside $u = g(s) = e^s$

Outside $f(u) = \csc u$

$$\frac{d}{ds}\csc(e^s)$$

$$= -\csc(e^s)\cot(e^s)\cdot e^s$$

$$= -e^s\csc(e^s)\cot(e^s)$$

$g'(s) = e^s$

$f'(u) = -\csc u \cot u$

$$\frac{d}{ds} f(g(s)) = f'(g(s))\,g'(s)$$

Example

Find the equation of the line tangent to the graph of $y = \cos^2 x$ at the point $(\frac{\pi}{4}, \frac{1}{2})$.

We need a point and a slope.

point: $(\pi/4, 1/2)$ given

slope: $m_{tan} = y'(c)$ here $c = \pi/4$

$y = (\cos x)^2$

inside function $u = g(x) = \cos x$

outside $f(u) = u^2$

$\frac{dy}{dx} = 2u \cdot (-\sin x)$

$g'(x) = -\sin x$, $f'(u) = 2u$

$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$

$$\frac{dy}{dx} = 2\cos x\,(-\sin x) = -2\cos x \sin x$$

$$m_{tan} = y'(\pi/4) = -2\cos\pi/4 \,\sin\pi/4 = -2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{-2}{2} = -1$$

point $(\underset{x_0}{\pi/4}, \underset{y_0}{1/2})$ slope $\underset{m}{m_{tan}} = -1$

$$y - y_0 = m(x - x_0)$$

$$y - \frac{1}{2} = -1(x - \pi/4)$$

$$\boxed{y = -x + \frac{\pi}{4} + \frac{1}{2}}$$

* We get the slope of the tangent line from the derivative

* Slope is a number (constant).

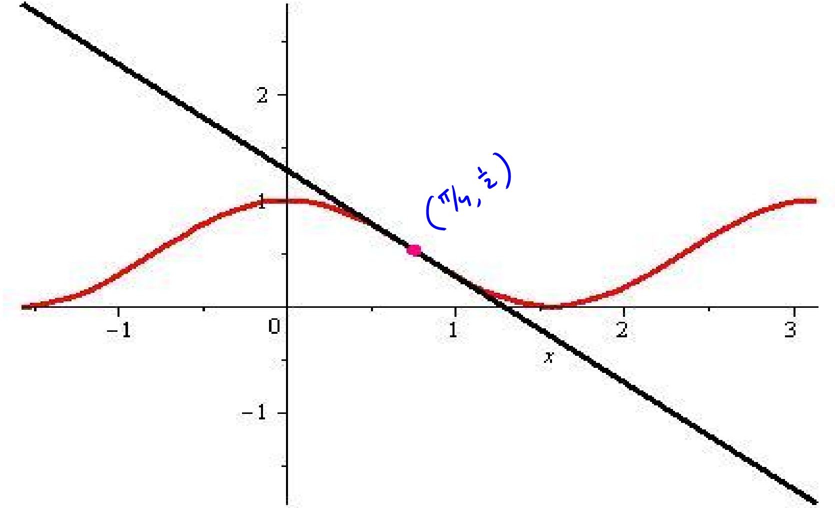

Figure: $y = \cos^2 x$ and the tangent line at $(\frac{\pi}{4}, \frac{1}{2})$.

We may be able to choose between differentiation methods.

Evaluate $\dfrac{d}{dx}\ \dfrac{\sin x}{x^3+2}$ using

quotient rule

$$\frac{d}{dx}\frac{f}{g} = \frac{f'g - fg'}{g^2}$$

(a) The quotient rule:

$$\frac{d}{dx}\frac{\sin x}{x^3+2} = \frac{\cos x\,(x^3+2) - \sin x\,(3x^2)}{(x^3+2)^2}$$

$$= \frac{(x^3+2)\cos x - 3x^2\sin x}{(x^3+2)^2}$$

(b) writing $\frac{\sin x}{x^3+2} = (\sin x)(x^3+2)^{-1}$ and using the chain rule.

$\boxed{\text{product rule} \quad \frac{d}{dx} fg = f'g + fg'}$

$\frac{d}{dx}(x^3+2)^{-1} = -u^{-2} \cdot (3x^2)$

$= -(x^3+2)^{-2}(3x^2)$

$= -3x^2(x^3+2)^{-2}$

If $u = g(x) = x^3+2$

$f(u) = u^{-1}$, then

$g'(x) = 3x^2$

$f'(u) = -1u^{-2} = -u^{-2}$

$\frac{d}{dx}\sin x\,(x^3+2)^{-1} = \cos x\,(x^3+2)^{-1} + \sin x\left(-3x^2(x^3+2)^{-2}\right)$

$$= \cos x \left(\frac{1}{x^3+2}\right) - 3x^2 \sin x \, \frac{1}{(x^3+2)^2}$$

$$= \frac{\cos x}{x^3+2} - \frac{3x^2 \sin x}{(x^3+2)^2}$$

$$= \frac{\cos x}{x^3+2} \cdot \left(\frac{x^3+2}{x^3+2}\right) - \frac{3x^2 \sin x}{(x^3+2)^2}$$

$$= \frac{\cos x \, (x^3+2)}{(x^3+2)^2} - \frac{3x^2 \sin x}{(x^3+2)^2}$$

$$= \frac{(x^3+2)\cos x - 3x^2 \sin x}{(x^3+2)^2}$$ as before

Questions

Evaluate $f'(x)$ where $f(x) = \sin(3x^2+2x)$

outside

inside

$u = g(x) = 3x^2+2x$

$f(u) = \sin u$

$g'(x) = 6x+2$

$f'(u) = \cos u$

(a) $f'(x) = \cos(3x^2+2x)$

(b) $f'(x) = (6x+2)\sin(x)+(3x^2+2x)\cos(x)$

(c) $f'(x) = -(6x+2)\cos(3x^2+2x)$

(d) $f'(x) = (6x+2)\cos(3x^2+2x)$

Questions

Find the equation of the line tangent to the graph of $f(x) = e^{\sin x}$ at the point $(0, f(0))$.

y-value for point $f(0) = e^{\sin 0} = e^0 = 1$

point $(0, 1)$

(a) $y = x + 1$

$m_{tan} = f'(0)$

Inside $g(x) = \sin x$

$f(u) = e^u$

(b) $y = 1$

$f'(x) = e^u \cdot \cos x$

$g'(x) = \cos x$

(c) $y = x - 1$

$= e^{\sin x} \cos x$

$f'(u) = e^u$

(d) $y = ex + 1$

$m_{tan} = f'(0) = e^{\sin 0} \cos 0 = 1 \cdot 1 = 1$