September 23 Math 2306 sec 51 Fall 2015

Section 4.3: Homogeneous Equations with Constant Coefficients

We are considering a second order, linear, homogeneous equation with constant coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.$$

Auxiliary a.k.a. Characteristic Equation

$$am^2 + bm + c = 0$$

The three cases to consider are (i) two distict real roots, (ii) one repeated real root, and (iii) a pair of complex conjugate roots.

Case I: Two distinct real roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac > 0$$

$$m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

A fundamental solution set is

$$y_1 = e^{m_1 x}, \qquad y_2 = e^{m_2 x}.$$

And the general solution is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

Example

Solve the IVP

$$y'' + y' - 12 = 0, \quad y(0) = 1, \quad y'(0) = 10$$

Characteristic eqn. $m^2 + m - 12 = 0$

$(m+4)(m-3) = 0 \implies$ $m = -4$ or $m = 3$

$y_1(x) = e^{-4x}$, $y_2(x) = e^{3x}$

General solution: $y = c_1 e^{-4x} + c_2 e^{3x}$

Apply the conditions $y(0)=1$ $\quad y'(0)=10$

$$y' = -4c_1 e^{-4x} + 3c_2 e^{3x}$$

$$y(0) = c_1 e^0 + c_2 e^0 = 1 \Rightarrow c_1 + c_2 = 1$$

$$y'(0) = -4c_1 e^0 + 3c_2 e^0 = 10 \Rightarrow -4c_1 + 3c_2 = 10$$

$$\begin{array}{r} 4c_1 + 4c_2 = 4 \\ -4c_1 + 3c_2 = 10 \\ \hline 7c_2 = 14 \end{array} \quad \text{add}$$

$$7c_2 = 14 \Rightarrow c_2 = 2$$

$$C_1 = 1 - C_2 = 1 - 2 = -1$$

The solution to the IVP is

$$\boxed{y = -e^{-4x} + 2e^{3x}.}$$

Case II: One repeated real root

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac = 0$$

$$y = c_1 e^{mx} + c_2 x e^{mx} \quad \text{where} \quad m = \frac{-b}{2a}$$

Use reduction of order to show that if $y_1 = e^{\frac{-bx}{2a}}$, then $y_2 = xe^{\frac{-bx}{2a}}$.

Standard form: $\quad y'' + \frac{b}{a}y' + \frac{c}{a}y = 0$

$$P(x) = \frac{b}{a}, \qquad -\int P(x)\,dx = -\int \frac{b}{a}\,dx = \frac{-b}{a}x$$

$$e^{-\int P(x)\,dx} = e^{-\frac{b}{a}x}$$

$$u = \int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx = \int \frac{e^{-\frac{b}{a}x}}{\left(e^{-\frac{b}{2a}x}\right)^2}\,dx$$

$$= \int \frac{e^{-\frac{b}{a}x}}{e^{-\frac{b}{a}x}}\,dx = \int dx = x$$

$$y_2 = u y_1 = x\,e^{-\frac{b}{2a}x}$$

Example

Solve the ODE

$$4y'' - 4y' + y = 0$$

Characteristic Equation:

$$4m^2 - 4m + 1 = 0$$

$$(2m-1)^2 = 0 \Rightarrow 2m - 1 = 0 \Rightarrow m = \frac{1}{2}$$

one repeated real root $m = \frac{1}{2}$.

$$y_1(x) = e^{\frac{1}{2}x} \quad \text{and} \quad y_2(x) = x e^{\frac{1}{2}x}$$

$$\boxed{y = C_1 e^{\frac{1}{2}x} + C_2 x e^{\frac{1}{2}x}}$$