September 23 Math 2306 sec 54 Fall 2015

Section 4.3: Homogeneous Equations with Constant Coefficients

We are considering a second order, linear, homogeneous equation with constant coefficients

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0.$$

Auxiliary a.k.a. Characteristic Equation

$$am^2+bm+c=0$$

The three cases to consider are (i) two distict real roots, (ii) one repeated real root, and (iii) a pair of complex conjugate roots.

Case I: Two distinct real roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac > 0$$

$$m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

A fundamental solution set is

$$y_1 = e^{m_1 x}, \qquad y_2 = e^{m_2 x}.$$

And the general solution is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

Example

Find the general solution of the ODE

$$y'' - 2y' - 2y = 0$$

Characteristic equation $\quad m^2 - 2m - 2 = 0$

$$m^2 - 2m + 1 - 1 - 2 = 0$$

$$(m-1)^2 - 3 = 0 \implies (m-1)^2 = 3$$

$$m - 1 = \pm\sqrt{3} \implies m = 1 \pm \sqrt{3}$$

$m_1 = 1+\sqrt{3}, \quad m_2 = 1-\sqrt{3}$

$y_1 = e^{(1+\sqrt{3})x}$ and $y_2 = e^{(1-\sqrt{3})x}$

The general solution is

$$\boxed{y = c_1 e^{(1+\sqrt{3})x} + c_2 e^{(1-\sqrt{3})x}.}$$

Example

Solve the IVP

$$y'' + y' - 12y = 0, \quad y(0) = 1, \quad y'(0) = 10$$

Characteristic equation : $m^2 + m - 12 = 0$

$$(m+4)(m-3) = 0$$

$$m_1 = -4 \quad \text{or} \quad m_2 = 3$$

so $\quad y_1 = e^{-4x} \quad$ and $\quad y_2 = e^{3x}$

The general solution to the DE is

$$y = c_1 e^{-4x} + c_2 e^{3x}$$

Apply $y(0) = 1$, $y'(0) = 10$

$$y' = -4c_1 e^{-4x} + 3c_2 e^{3x}$$

$$y(0) = c_1 e^0 + c_2 e^0 = 1 \quad \Longrightarrow \quad c_1 + c_2 = 1$$

$$y'(0) = -4c_1 e^0 + 3c_2 e^0 = 10 \quad \Longrightarrow \quad -4c_1 + 3c_2 = 10$$

$$4c_1 + 4c_2 = 4$$

$$\underline{-4c_1 + 3c_2 = 10} \quad \text{add}$$

$$7c_2 = 14 \Rightarrow c_2 = 2$$

$$c_1 = -1$$

$$c_1 = 1 - c_2 = 1 - 2 = -1$$

$$c_2 = 2$$

$$\boxed{\text{The solution to the IVP is} \quad y = -e^{-4x} + 2e^{3x}.}$$

Case II: One repeated real root

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac = 0$$

$$y = c_1 e^{mx} + c_2 x e^{mx} \quad \text{where} \quad m = \frac{-b}{2a}$$

Use reduction of order to show that if $y_1 = e^{\frac{-bx}{2a}}$, then $y_2 = xe^{\frac{-bx}{2a}}$.

Standard form $\quad y'' + \frac{b}{a}y' + \frac{c}{a}y = 0$

$P(x) = \frac{b}{a}$, $\quad -\int P(x)\,dx = -\int \frac{b}{a}\,dx = \frac{-b}{a}x$

$$e^{-\int P(x)\,dx} = e^{\frac{-b}{a}x}$$

$$u = \int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx = \int \frac{e^{-\frac{b}{a}x}}{\left(e^{-\frac{b}{2a}x}\right)^2}\,dx = \int \frac{e^{-\frac{b}{a}x}}{e^{-\frac{b}{a}x}}\,dx$$

$$= \int dx = x$$

so $\quad y_2 = u y_1 = x\, e^{-\frac{b}{2a}x}$

Example

Solve the ODE

$$4y'' - 4y' + y = 0$$

Characteristic Eqn: $4m^2 - 4m + 1 = 0$

$(2m-1)^2 = 0$

$2m - 1 = 0 \Rightarrow m = \frac{1}{2}$

$y_1 = e^{\frac{1}{2}x}$ and $y_2 = xe^{\frac{1}{2}x}$

The general solution is

$$y = C_1 e^{\frac{1}{2}x} + C_2 x e^{\frac{1}{2}x}$$

Example

Solve the IVP

$$y'' + 6y' + 9y = 0, \quad y(0) = 4, \quad y'(0) = 0$$

Characteristic Eqn: $m^2 + 6m + 9 = 0$

$$(m+3)^2 = 0$$

$$m = -3$$

$$y_1(x) = e^{-3x} \quad \text{and} \quad y_2(x) = x e^{-3x}$$

The general solution to the ODE is

$$y = c_1 e^{-3x} + c_2 x e^{-3x}.$$

Apply $y(0) = 4 \qquad y'(0) = 0$

$$y' = -3c_1 e^{-3x} + c_2 e^{-3x} - 3c_2 x e^{-3x}$$

$$y(0) = c_1 e^{0} + c_2 \cdot 0 e^{0} = 4 \implies c_1 = 4$$

$$y'(0) = -3(4)e^0 + c_2 e^0 - 3c_2 \cdot 0 e^0 = 0$$

$$-12 + c_2 = 0 \implies c_2 = 12$$

The solution to the IVP is

$$\boxed{y = 4e^{-3x} + 12x e^{-3x}.}$$

Case III: Complex conjugate roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac < 0$$

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x)), \quad \text{where the roots}$$

$$m = \alpha \pm i\beta, \quad \alpha = \frac{-b}{2a} \quad \text{and} \quad \beta = \frac{\sqrt{4ac - b^2}}{2a}$$

The solutions can be written as

$$Y_1 = e^{(\alpha + i\beta)x} = e^{\alpha x} e^{i\beta x}, \quad \text{and} \quad Y_2 = e^{(\alpha - i\beta)x} = e^{\alpha x} e^{-i\beta x}.$$

Deriving the solutions Case III

Recall Euler's Formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$y_1 = e^{(\alpha+i\beta)x} = e^{\alpha x}e^{i\beta x} = e^{\alpha x}\left(\cos(\beta x) + i\sin(\beta x)\right)$$

$$y_2 = e^{(\alpha-i\beta)x} = e^{\alpha x}e^{-i\beta x} = e^{\alpha x}\left(\cos(\beta x) - i\sin(\beta x)\right)$$

We'll use the principle of super position to eliminate the complex stuff.

$$\Psi_1(x) = e^{\alpha x}\cos(\beta x) + i\, e^{\alpha x}\sin(\beta x)$$

$$\Psi_2(x) = e^{\alpha x}\cos(\beta x) - i\, e^{\alpha x}\sin(\beta x)$$

Set $y_1(x) = \frac{1}{2}\left(\Psi_1(x) + \Psi_2(x)\right)$

$$= \frac{1}{2}\left(2e^{\alpha x}\cos(\beta x)\right) = e^{\alpha x}\cos(\beta x)$$

$$y_2(x) = \frac{1}{2i}\left(\Psi_1(x) - \Psi_2(x)\right)$$

$$= \frac{1}{2i}\left(2i\, e^{\alpha x}\sin(\beta x)\right) = e^{\alpha x}\sin(\beta x)$$

So our fundamental solution set

$$y_1(x) = e^{\alpha x}\cos(\beta x), \quad y_2(x) = e^{\alpha x}\sin(\beta x)$$