Sept 24 Math 2306 sec. 53 Fall 2018

Section 8: Homogeneous Equations with Constant Coefficients

We consider a second order, linear, homogeneous equation with constant coefficients

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0.$$

Question: What sort of function y could be expected to satisfy

$$y''=\text{constant}\,y'+\text{constant}\,y?$$

We look for solutions of the form $y = e^{mx}$ with m constant.

Sub into $ay'' + by' + cy = 0$

If $y = e^{mx}$, then $y' = me^{mx}$ and $y'' = m^2 e^{mx}$

$$ay'' + by' + cy = a(m^2 e^{mx}) + b(me^{mx}) + ce^{mx} = 0$$

$$e^{mx}(am^2 + bm + c) = 0$$

This product is zero if $\boxed{am^2 + bm + c = 0}$.

So we have solutions to the ODE of the form e^{mx} if m solves the quadratic equation.

Auxiliary a.k.a. Characteristic Equation

$$am^2 + bm + c = 0$$

There are three cases:

I $b^2 - 4ac > 0$ and there are two distinct real roots $m_1 \neq m_2$

II $b^2 - 4ac = 0$ and there is one repeated real root $m_1 = m_2 = m$

III $b^2 - 4ac < 0$ and there are two roots that are complex conjugates $m_{1,2} = \alpha \pm i\beta$

Case I: Two distinct real roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac > 0$$

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} \quad \text{where} \quad m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Show that $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$ are linearly independent.

Let's use the Wronskian.

$$W(y_1, y_2)(x) = \begin{vmatrix} e^{m_1 x} & e^{m_2 x} \\ m_1 e^{m_1 x} & m_2 e^{m_2 x} \end{vmatrix}$$

$$= e^{m_1 x}\left(m_2 e^{m_2 x}\right) - m_1 e^{m_1 x}\left(e^{m_2 x}\right)$$

$$= e^{(m_1+m_2)x}\left(m_2 - m_1\right)$$

$e^{(m_1+m_2)x} \neq 0$. $m_2 - m_1 \neq 0$

since $m_1 \neq m_2$

two distinct real root case

y_1 and y_2 are linearly independent since

$W \neq 0$.

Example

Solve the IVP

$$y'' + y' - 12y = 0, \quad y(0) = 1, \quad y'(0) = 10$$

The characteristic equation is

$$m^2 + m - 12 = 0$$

$$(m+4)(m-3) = 0 \Rightarrow$$

2 roots $m_1 = -4$ and $m_2 = 3$

$y_1 = e^{-4x}$ $y_2 = e^{3x}$

$$y = C_1 e^{-4x} + C_2 e^{3x}$$

Apply the IC

$$y' = -4C_1 e^{-4x} + 3C_2 e^{3x}$$

$y(0) = 1 = c_1 e^0 + c_2 e^0 \Rightarrow$ $c_1 + c_2 = 1$ times 4

$y'(0) = 10 = -4c_1 e^0 + 3c_2 e^0 \Rightarrow$ $-4c_1 + 3c_2 = 10$

$4c_1 + 4c_2 = 4$

$7c_2 = 14$

$c_2 = 2$

$c_1 = 1 - c_2 = 1 - 2 = -1$

The solution to the IVP is

$$\boxed{y = -e^{-4x} + 2e^{3x}}$$

Case II: One repeated real root

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac = 0$$

$$y = c_1 e^{mx} + c_2 x e^{mx} \quad \text{where} \quad m = \frac{-b}{2a}$$

Use reduction of order to show that if $y_1 = e^{\frac{-bx}{2a}}$, then $y_2 = xe^{\frac{-bx}{2a}}$.

$y_2 = u y_1$ where $u = \displaystyle\int \frac{e^{-\int P(x)dx}}{(y_1)^2}\,dx$

Standard form: $y'' + \frac{b}{a} y' + \frac{c}{a} y = 0$

$P(x) = \frac{b}{a}$ so $-\int P(x)\,dx = -\int \frac{b}{a}\,dx = \frac{-b}{a}x$

$$(y_1)^2 = \left(e^{\frac{-b}{2a}x}\right)^2 = e^{2\left(\frac{-b}{2a}x\right)} = e^{\frac{-b}{a}x}$$

$$u = \int \frac{e^{-\frac{b}{a}x}}{e^{\frac{-b}{a}x}}\,dx = \int dx = x$$

So $\quad y_2 = uy_1 = x\,e^{\frac{-b}{2a}x}$

Example

Solve the ODE

$$4y'' - 4y' + y = 0$$

Characteristic eqn $4m^2 - 4m + 1 = 0$

$(2m-1)^2 = 0$

$m = \frac{1}{2}$ repeated root

$y_1 = e^{\frac{1}{2}x}, \quad y_2 = xe^{\frac{1}{2}x}$

The general solution is

$$y = c_1 e^{\frac{1}{2}x} + c_2 x e^{\frac{1}{2}x}.$$