September 25 Math 2306 sec 51 Fall 2015

Section 4.3: Homogeneous Equations with Constant Coefficients

We are considering a second order, linear, homogeneous equation with constant coefficients

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0.$$

Auxiliary a.k.a. Characteristic Equation

$$am^2+bm+c=0$$

The three cases to consider are (i) two distict real roots, (ii) one repeated real root, and (iii) a pair of complex conjugate roots.

Case I: Two distinct real roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac > 0$$

$$m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

A fundamental solution set is

$$y_1 = e^{m_1 x}, \qquad y_2 = e^{m_2 x}.$$

And the general solution is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

Case II: One repeated real root

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac = 0$$

$$m_1 \ = \ m_2 \ = \ m = \frac{-b}{2a}$$

A fundamental solution set is

$$y_1 = e^{mx}, \qquad y_2 = xe^{mx}.$$

And the general solution is

$$y = c_1 e^{mx} + c_2 xe^{mx}.$$

Case III: Complex conjugate roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac < 0$$

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x)), \quad \text{where the roots}$$

$$m = \alpha \pm i\beta, \quad \alpha = \frac{-b}{2a} \quad \text{and} \quad \beta = \frac{\sqrt{4ac - b^2}}{2a}$$

The solutions can be written as

$$Y_1 = e^{(\alpha + i\beta)x} = e^{\alpha x} e^{i\beta x}, \quad \text{and} \quad Y_2 = e^{(\alpha - i\beta)x} = e^{\alpha x} e^{-i\beta x}.$$

Principle of Superposition

Deriving the solutions Case III

Recall Euler's Formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$Y_1(x) = e^{(\alpha+i\beta)x} = e^{\alpha x} \cdot e^{i\beta x} = e^{\alpha x}\left(\cos(\beta x) + i\sin(\beta x)\right)$$

$$Y_2(x) = e^{(\alpha-i\beta)x} = e^{\alpha x}\, e^{-i\beta x} = e^{\alpha x}\left(\cos(\beta x) - i\sin(\beta x)\right)$$

$$Y_1 = e^{\alpha x}\cos\beta x + i\, e^{\alpha x}\sin\beta x$$

$$Y_2 = e^{\alpha x}\cos\beta x - i\, e^{\alpha x}\sin\beta x$$

Let $$y_1 = \frac{1}{2}(Y_1 + Y_2) = \frac{1}{2}\left(2e^{\alpha x}\cos\beta x\right)$$

$$y_1 = e^{\alpha x}\cos\beta x$$

$$y_2 = \frac{1}{2i}(Y_1 - Y_2) = \frac{1}{2i}\left(2i\, e^{\alpha x}\sin\beta x\right)$$

$$y_2 = e^{\alpha x}\sin\beta x$$

So a fundamental solution set is

$$e^{\alpha x}\cos\beta x, \quad e^{\alpha x}\sin\beta x$$

And the general solution is

$$y = C_1 e^{\alpha x} \cos\beta x + C_2 e^{\alpha x} \sin\beta x$$

Example

Solve the ODE

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 6x = 0$$

Characteristic Eqn: $m^2 + 4m + 6 = 0$

$$m^2 + 4m + 4 - 4 + 6 = 0$$

$$m^2 + 4m + 4 + 2 = 0 \Rightarrow (m+2)^2 + 2 = 0$$

$$(m+2)^2 = -2 \Rightarrow m+2 = \pm\sqrt{2}\,i$$

$$m = -2 \pm \sqrt{2}\,i$$

$\alpha = -2$ and $\beta = \sqrt{2}$

$$x_1 = e^{-2t}\cos(\sqrt{2}\,t)\ , \quad x_2 = e^{-2t}\sin(\sqrt{2}\,t)$$

The general solution is

$$x = c_1 e^{-2t}\cos(\sqrt{2}\,t) + c_2 e^{-2t}\sin(\sqrt{2}\,t)$$

Example

Solve the IVP

$$y'' + 4y = 0, \quad y(0) = 3, \quad y'(0) = -5$$

Characteristic Eqn: $m^2 + 4 = 0$

$m^2 = -4$

$m = \pm 2i = 0 \pm 2i$

$\alpha = 0, \quad \beta = 2$

$y_1 = e^{0x} \cos 2x = \cos 2x, \quad y_2 = e^{0x} \sin 2x = \sin 2x$

The general solution is

$$y = c_1 \cos 2x + c_2 \sin 2x$$

Impose the conditions $y(0) = 3$, $y'(0) = -5$

$$y' = -2c_1 \sin 2x + 2c_2 \cos 2x$$

$$y(0) = c_1 \cos 0 + c_2 \sin 0 = 3 \implies c_1 = 3$$

$$y'(0) = -2c_1 \sin 0 + 2c_2 \cos 0 = -5$$

$$2c_2 = -5 \implies c_2 = \frac{-5}{2}$$

The solution to the IVP is

$$y = 3\cos 2x - \frac{5}{2}\sin 2x$$

Higer Order Linear Constant Coefficient ODEs

- The same approach applies. For an n^{th} order equation, we obtain an n^{th} degree polynomial.
- Complex roots must appear in conjugate pairs (due to real coefficients) giving a pair of solutions $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$.
- If a root m is repeated k times, we get k linearly independent solutions

$$e^{mx}, \quad xe^{mx}, \quad x^2e^{mx}, \quad \ldots, \quad x^{k-1}e^{mx}$$

or in conjugate pairs cases $2k$ solutions

$$e^{\alpha x}\cos(\beta x),\ e^{\alpha x}\sin(\beta x), \quad xe^{\alpha x}\cos(\beta x),\ xe^{\alpha x}\sin(\beta x), \ldots,$$
$$x^{k-1}e^{\alpha x}\cos(\beta x),\ x^{k-1}e^{\alpha x}\sin(\beta x)$$

- It may require a computer algebra system to find the roots for a high degree polynomial.

Example

Solve the ODE

$y''' - 4y' = 0$

This is 3rd order linear, homogeneous, constant coefficient.

Characteristic Eqn:

$$m^3 - 4m = 0 \Rightarrow m(m^2-4) = 0$$

$$m(m-2)(m+2) = 0$$

$$m = 0, \quad m = 2, \quad \text{or} \quad m = -2$$

$$m_1 = 0, \quad m_2 = 2, \quad m_3 = -2$$

$$y_1 = e^{m_1 x} = e^{0x} = 1, \quad y_2 = e^{m_2 x} = e^{2x}$$

$$y_3 = e^{m_3 x} = e^{-2x}$$

The general solution is

$$y = C_1(1) + C_2 e^{2x} + C_3 e^{-2x}$$

$$y = C_1 + C_2 e^{2x} + C_3 e^{-2x}$$

Example

Solve the ODE

$y''' - 3y'' + 3y' - y = 0$

Characteristic Eqn:

$$m^3 - 3m^2 + 3m - 1 = 0$$

$$(m-1)^3 = 0$$

$m = 1$ is a triple root.

So $y_1 = e^{mx} = e^x$

$y_2 = xy_1 = xe^x$ and $y_3 = xy_2 = x^2 e^x$

The general solution is

$$y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x.$$