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Section 6: Linear Equations Theory and Terminology

We're still considering this equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

with the assumptions $a_n(x) \neq 0$ and $a_i(x)$ are continuous on I.

General Solution

Let $y_1, y_2, \ldots, y_n$ be a fundamental solution set of the nth order linear homogeneous equation. Then the **general solution** of the equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Example

Verify that $y_1 = x^2$ and $y_2 = x^3$ form a fundamental solution set of the ODE

$$x^2y'' - 4xy' + 6y = 0 \quad \text{on} \quad (0, \infty),$$

and determine the general solution.

First note that (ii) there are two of them and (iii) we showed that they are linearly independent.

(i) We have to verify that y_1 and y_2 are actually solutions.

Verify y_1: $y_1 = x^2$, $y_1' = 2x$, $y_1'' = 2$

$$x^2y_1'' - 4xy_1' + 6y_1 = x^2(2) - 4x(2x) + 6(x^2)$$
$$= 2x^2 - 8x^2 + 6x^2 = 0$$

y_1 is a solution.

Verify y_2: $y_2 = x^3$, $y_2' = 3x^2$, $y_2'' = 6x$

$$x^2 y_2'' - 4x y_2' + 6y_2 = x^2(6x) - 4x(3x^2) + 6x^3$$

$$= 6x^3 - 12x^3 + 6x^3 = 0$$

We have two linearly independent solutions,

hence y_1, y_2 are a fundamental solution set.

The general solution

$$y = c_1 y_1 + c_2 y_2$$

$$y = c_1 x^2 + c_2 x^3$$

Nonhomogeneous Equations

Now we will consider the equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

where g is not the zero function. We'll continue to assume that a_n doesn't vanish and that a_i and g are continuous.

The **associated homogeneous equation** is

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0.$$

The associated homogeneous equation has the same left hand side, it just has zero on the right side.

Theorem: General Solution of Nonhomogeneous Equation

Let y_p be any solution of the nonhomogeneous equation, and let y_1, $y_2, \ldots, y_n$ be any fundamental solution set of the associated homogeneous equation.

Then the general solution of the nonhomogeneous equation is

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p(x)$$

where c_1, $c_2, \ldots, c_n$ are arbitrary constants.

Note the form of the solution $y_c + y_p$!
The complementary part $y_c = c_1 y_1 + \cdots + c_n y_n$.

Principle of Superposition: Nonhomogeneous Equations

Suppose we have a nonhomogeneous, linear equation of the form

$$a_n(x)\frac{d^n y}{dx^n} + \cdots + a_0(x)y = g_1(x) + g_2(x) \tag{1}$$

and suppose that y_{p_1} solves

$$a_n(x)\frac{d^n y}{dx^n} + \cdots + a_0(x)y = g_1(x) \quad \text{and}$$

y_{p_2} solves

$$a_n(x)\frac{d^n y}{dx^n} + \cdots + a_0(x)y = g_2(x).$$

Then a particular solution to (1) is

$$y_p = y_{p_1}(x) + y_{p_2}(x).$$

The Principle of Superposition for Nonhomogeneous Equations

Let $y_{p_1}, y_{p_2}, \ldots, y_{p_k}$ be k particular solutions to the nonhomogeneous linear equations

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g_i(x)$$

for $i = 1, \ldots, k$. Assume the domain of definition for all k equations is a common interval I.

Then

$$y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_k}$$

is a particular solution of the nonhomogeneous equation

$$a_n(x)\frac{d^n y}{dx^n} + \cdots + a_0(x)y = g_1(x) + g_2(x) + \cdots + g_k(x).$$

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

We will construct the general solution by considering sub-problems.
We'll consider the right hand side the sum

$$g_1(x) + g_2(x) \quad \text{where} \quad g_1(x) = 36 \quad \text{and} \quad g_2(x) = -14x.$$

(a) **Part 1** Verify that

$$y_{p_1} = 6 \quad \text{solves} \quad x^2y'' - 4xy' + 6y = 36.$$

$y_{p_1}' = 0$, $y_{p_1}'' = 0$

$$x^2y_{p_1}'' - 4xy_{p_1}' + 6y_{p_1} \stackrel{?}{=} 36$$

$$x^2(0) - 4x(0) + 6(6) \stackrel{?}{=} 36$$

$$36 = 36$$

y_{p_1} solves the ODE

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(b) **Part 2** Verify that

$$y_{p_2} = -7x \quad \text{solves} \quad x^2y'' - 4xy' + 6y = -14x.$$

$y_{p_2}' = -7$, $y_{p_2}'' = 0$

$$x^2 y_{p_2}'' - 4x y_{p_2}' + 6y_{p_2} \stackrel{?}{=} -14x$$

$$x^2(0) - 4x(-7) + 6(-7x) \stackrel{?}{=} -14x$$

$$28x - 42x \stackrel{?}{=} -14x$$

$$-14x = -14x$$

y_{p_2} is a solution.

Example $x^2y'' - 4xy' + 6y = 36 - 14x$

(c) **Part 3** We already know that $y_1 = x^2$ and $y_2 = x^3$ is a fundamental solution set of

$$x^2y'' - 4xy' + 6y = 0.$$

Use this along with results (a) and (b) to write the general solution of $x^2y'' - 4xy' + 6y = 36 - 14x$.

The complementary solution $y_c = c_1x^2 + c_2x^3$

By the principle of superposition $y_p = y_{p_1} + y_{p_2}$

$y_p = 6 - 7x$

The general solution is $y = c_1x^2 + c_2x^3 + 6 - 7x$

Solve the IVP

$$x^2y'' - 4xy' + 6y = 36 - 14x, \quad y(1) = 0, \quad y'(1) = -5$$

The general solution to the ODE is

$$y = c_1 x^2 + c_2 x^3 + 6 - 7x$$

we have to impose the initial conditions.

$$y' = 2c_1 x + 3c_2 x^2 - 7$$

$$y(1) = c_1(1^2) + c_2(1^3) + 6 - 7(1) = 0 \Rightarrow c_1 + c_2 = 1$$

$$y'(1) = 2c_1(1) + 3c_2(1^2) - 7 = -5 \qquad 2c_1 + 3c_2 = 2$$

$$2c_1 + 2c_2 = 2$$

Subtract $\quad 2c_1 + 3c_2 = 2$

$$-c_2 = 0 \Rightarrow c_2 = 0 \qquad c_1 = 1$$

The solution to the IVP is

$$y = x^2 + 6 - 7x$$

Section 7: Reduction of Order

We'll focus on second order, linear, homogeneous equations. Recall that such an equation has the form

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0.$$

Let us assume that $a_2(x) \neq 0$ on the interval of interest. We will write our equation in **standard form**

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

where $P = a_1/a_2$ and $Q = a_0/a_2$.

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

Recall that every fundmantal solution set will consist of two linearly independent solutions y_1 and y_2, and the general solution will have the form

$$y = c_1 y_1(x) + c_2 y_2(x).$$

Suppose we happen to know one solution $y_1(x)$. **Reduction of order** is a method for finding a second linearly independent solution $y_2(x)$ that starts with the assumption that

u - unknown

$$y_2(x) = u(x)y_1(x)$$ y_1 is known.

for some function $u(x)$. The method involves finding the function u.

$U(x)$ can't be constant due to linear independence.

Example

Verify that $y_1 = e^{-x}$ is a solution of $y'' - y = 0$. Then find a second solution y_2 of the form

$$y_2(x) = u(x)y_1(x) = e^{-x}u(x).$$

Confirm that the pair y_1, y_2 is linearly independent.

Verify y_1 solves the ODE: $y_1' = -e^{-x}$, $y_1'' = e^{-x}$

$y_1'' - y_1 = e^{-x} - e^{-x} = 0$, y_1 is a solution.

$y_2 = uy_1$ is a solution, so substitute into the ODE.

$y_2 = ue^{-x}$, $y_2' = u'e^{-x} - ue^{-x}$

$y_2'' = u''e^{-x} - u'e^{-x} - u'e^{-x} + ue^{-x} = u''e^{-x} - 2u'e^{-x} + ue^{-x}$

Plugging in $y_2'' - y_2 =$

$$u''e^{-x} - 2u'e^{-x} + ue^{-x} - ue^{-x} = 0$$

$$u''e^{-x} - 2u'e^{-x} = 0$$

Divide by e^{-x} to get

$$u'' - 2u' = 0$$

Set $w = u'$, so $w' = u''$ and w satisfies

$$w' - 2w = 0 \quad \text{a } 1^{st} \text{ order ODE}$$

Solve by separation of variables

$$\frac{dw}{dx} = 2w$$

$$\int \frac{1}{w}\, dw = \int 2\, dx \qquad \text{assuming } w > 0$$

$$\ln w = 2x$$

$$w = e^{2x}$$

$u' = w$ so $u = \int w\, dx = \int e^{2x}\, dx = \frac{1}{2} e^{2x}$

So $y_2 = u y_1 = \frac{1}{2} e^{2x}\left(e^{-x}\right) = \frac{1}{2} e^{x}$

we can take $y_2 = 2\left(\frac{1}{2}e^x\right) = e^x$

Let's verify that $y_1 = e^{-x}$, $y_2 = e^x$ are linearly independent. We'll use the Wronskian

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^x \\ -e^{-x} & e^x \end{vmatrix}$$

$$= e^{-x}\left(e^x\right) - \left(-e^{-x}\right)e^x = 1 - (-1) = 2$$

$W \neq 0$, they are linearly independent.

The general solution to the ODE is

$$y = C_1 e^{-x} + C_2 e^{x}$$