Sept. 26 Math 1190 sec. 51 Fall 2016

Section 3.1: The Chain Rule

The chain rule can be iterated to account for multiple compositions. For example, suppose f, g, h are appropriately differentiable, then

$$\frac{d}{dx}(f \circ g \circ h)(x) = \frac{d}{dx} f(g(h(x)) = f'(g(h(x))\, g'(h(x))\, h'(x)$$

Note that the outermost function is f, and its inner function is a composition $g(h(x))$.

So the derivative of the outer function evaluated at the inner is $f'(g(h(x))$ which is multiplied by the derivative of the inner function—**itself based on the chain rule**—$g'(h(x))h'(x)$.

If $h(x) = v$, $g(v) = u$, and $y = f(u)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}$$

Example

$\exp(u) = e^u$

Evaluate the derivative $\dfrac{d}{dx} \exp(\cot(\pi x))$

$\exp(\cot(\pi x)) = f(g(h(x)))$, f

$h(x) = \pi x = v$

$g(v) = \cot v = u$

$f(u) = e^u$

$\dfrac{d}{dx} \exp(\cot(\pi x))$

$= e^u \cdot (-\csc^2 v) \cdot \pi$

$= e^{\cot v} (-\csc^2 v) \cdot \pi$

$= -\csc^2(\pi x)\, e^{\cot(\pi x)} \cdot \pi$

$h'(x) = \pi$

$g'(v) = -\csc^2 v$

$f'(u) = e^u$

$$= -\pi \csc^2(\pi x)\, e^{\cot(\pi x)}$$

Exponential of Base a

Let $a > 0$ with $a \neq 1$. By properties of logs and exponentials

$$a^x = e^{(\ln a)x}.$$

$(\ln a)x$ is the constant $\ln a$ times the variable x.

If $g(x) = (\ln a)x$, $g'(x) = \ln a$
$f(u) = e^u$, $f'(u) = e^u$

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{(\ln a)x} = e^{(\ln a)x} \cdot \ln a = a^x \ln a = (\ln a)\, a^x$$

Theorem: (Derivative of $y = a^x$) Let $a > 0$ and $a \neq 1$. Then

$$\frac{d}{dx} a^x = a^x \ln a$$

Example

$\frac{d}{dx} a^x = a^x \ln a$

Evaluate

(a) $\dfrac{d}{dx}4^x$ $= 4^x \ln 4$

(b) $\dfrac{d}{dx}2^{\cos x}$ $= \left(2^u \ln 2\right)(-\sin x)$

$= 2^{\cos x}(\ln 2)(-\sin x)$

$= -(\ln 2)\sin x \; 2^{\cos x}$

If $u = g(x) = \cos x$

$g'(x) = -\sin x$

$f(u) = 2^u$

$f'(u) = 2^u \ln 2$

Section 3.2: Implicit Differentiation; Derivatives of the Inverse Trigonometric Functions

The chain rule states that for a differentiable composition $f(g(x))$

$$\frac{d}{dx}\,f(g(x)) = f'(g(x))g'(x)$$

For $y = f(u)$ and $u = g(x)$

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$$

Example

Assume f is a differentiable function of x. Find an expression for the derivative:

$$\frac{d}{dx}\,(f(x))^2 = 2f(x)\cdot f'(x)$$

Inside is $f(x)$

outside is u^2

$$\frac{d}{dx}\,\tan(f(x)) = \sec^2(f(x))\cdot f'(x)$$

Inside $f(x)$

outside is $\tan(u)$

Example

Suppose we know that $y = f(x)$ for some differentiable function (but we don't know exactly what f is). Find an expression for the derivative.

$$\frac{d}{dx} \; y^3 = 3y^2 \cdot \frac{dy}{dx}$$

Inside is y

outside is u^3

"3 times y^2 times $\frac{dy}{dx}$"

$$\frac{d}{dx} \; x^2y^2 = \left(\frac{d}{dx}x^2\right)y^2 + x^2\left(\frac{d}{dx}y^2\right)$$

$$= 2xy^2 + x^2\left(2y \cdot \frac{dy}{dx}\right)$$

$$= 2xy^2 + 2x^2y\frac{dy}{dx}$$

product

x^2 times y^2

product rule

Product rule

$$\frac{d}{dx}fg = f'g + fg'$$

Implicitly defined functions

A relation—an equation involving two variables x and y—such as

$$x^2 + y^2 = 16 \quad \text{or} \quad (x^2 + y^2)^3 = x^2$$

implies that y is defined to be one or more functions of x.

e.g. $x^2+y^2=16 \Rightarrow y^2 = 16-x^2$

$\Rightarrow y = \sqrt{16-x^2}$ OR $y = -\sqrt{16-x^2}$

2 possible functions

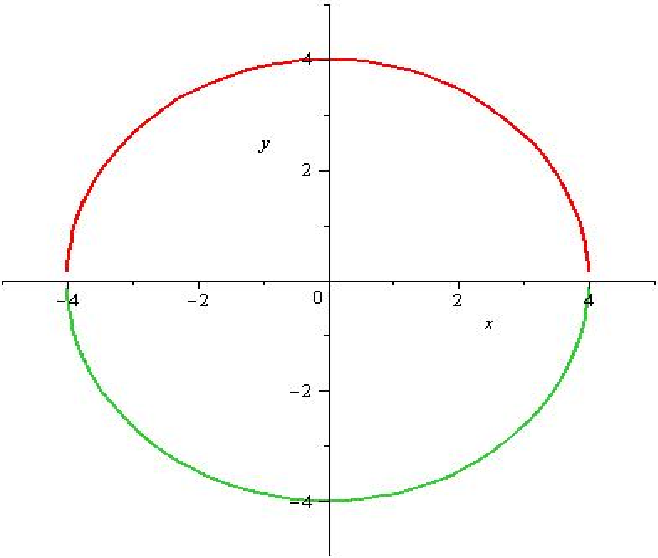

Figure: $x^2 + y^2 = 16$

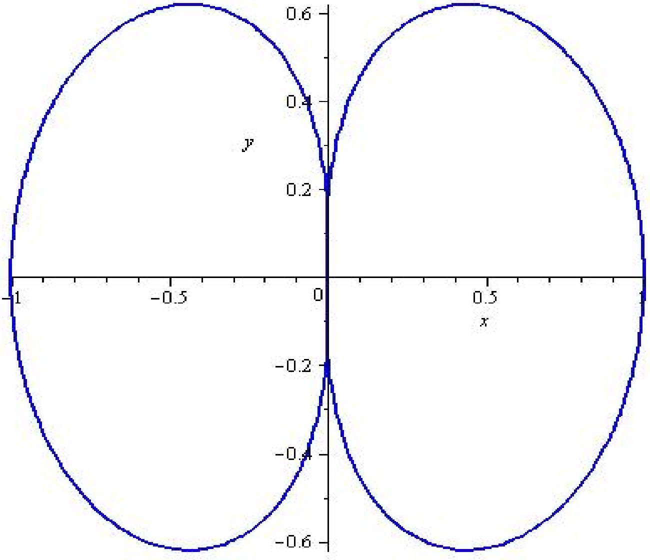

Figure: $(x^2 + y^2)^3 = x^2$

Explicit -vs- Implicit

A function is defined **explicitly** when given in the form

$$y = f(x).$$

e.g. $y = \tan(x^2)$ or $y = e^{2x}$

A function is defined *implicitly* when it is given as a relation

$$F(x, y) = C,$$

for constant C.

e.g. $x^2 + y^2 = 16$, $x^2y^2 + \tan(xy) = 15$

Implicit Differentiation

Since $x^2 + y^2 = 16$ *implies* that y is a function of x, we can consider it's derivative.

$$\text{Find } \frac{dy}{dx} \text{ given } \quad x^2 + y^2 = 16.$$

Start w/ the relation, take the derivative $\frac{d}{dx}$, with respect to x, of both sides.

$$\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}(16)$$

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = 0$$

$$2x + 2y \cdot \frac{dy}{dx} = 0$$

Chain rule

$$\frac{d}{dx}y^2 = 2y \cdot \frac{dy}{dx}$$

y - inside

u^2 - outside

Now solve for $\frac{dy}{dx}$

Note

$\frac{d}{dx}x^2 = 2x \cdot \frac{dx}{dx}$

$= 2x \cdot 1$

$= 2x$

$2y\frac{dy}{dx} = -2x$ Subtract $2x$

$\frac{dy}{dx} = \frac{-2x}{2y}$ Divide by $2y$

$\Rightarrow \boxed{\frac{dy}{dx} = \frac{-x}{y}}$

Show that the same result is obtained knowing

$$y = \sqrt{16 - x^2} \quad \text{or} \quad y = -\sqrt{16 - x^2}.$$

Square root rule: $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$

$$\frac{d}{dx}y = \frac{d}{dx}\sqrt{16 - x^2} = \frac{1}{2\sqrt{16-x^2}} \cdot (0-2x)$$

inside $16 - x^2$

outside $\sqrt{u}$

$$= \frac{-2x}{2\sqrt{16-x^2}} = \frac{-x}{\sqrt{16-x^2}} = \frac{-x}{y}$$

since $y = \sqrt{16-x^2}$

If $y = -\sqrt{16 - x^2}$ then

$$\frac{d}{dx} y = \frac{d}{dx}\left(-\sqrt{16 - x^2}\right) = -\frac{d}{dx}\sqrt{16 - x^2}$$

$$= -\left(\frac{-x}{\sqrt{16 - x^2}}\right)$$

$$= \frac{-x}{-\sqrt{16 - x^2}} = \frac{-x}{y}$$

Example

Find $\dfrac{dy}{dx}$ given $x^2 - 3xy + y^2 = y.$

Take $\frac{d}{dx}$ of both sides.

$$\frac{d}{dx}\left(x^2 - 3xy + y^2\right) = \frac{d}{dx}\, y$$

$$\frac{d}{dx}\, x^2 - 3\,\frac{d}{dx}(xy) + \frac{d}{dx}\, y^2 = \frac{dy}{dx}$$

product

$$2x - 3\left(1 \cdot y + x\,\frac{dy}{dx}\right) + 2y \cdot \frac{dy}{dx} = \frac{dy}{dx}$$

isolate $\frac{dy}{dx}$ using algebra

$$2x - 3y - 3x\frac{dy}{dx} + 2y\frac{dy}{dx} = \frac{dy}{dx}$$

move $\frac{dy}{dx}$ terms to the left and all others to the right

$$2x - 3y - 3x\frac{dy}{dx} + 2y\frac{dy}{dx} - \frac{dy}{dx} = 0$$

$$-3x\frac{dy}{dx} + 2y\frac{dy}{dx} - \frac{dy}{dx} = 3y - 2x$$

$$\frac{dy}{dx}(-3x + 2y - 1) = 3y - 2x$$

$$\frac{dy}{dx} = \frac{3y - 2x}{-3x + 2y - 1} = \frac{3y - 2x}{2y - 3x - 1}$$

Finding a Derivative Using Implicit Differentiation:

- Take the derivative of both sides of an equation with respect to the independent variable.
- Use all necessary rules for differenting powers, products, quotients, trig functions, exponentials, compositions, etc.
- Remember the chain rule for each term involving the dependent variable (e.g. mult. by $\frac{dy}{dx}$ as required).
- Use necessary algebra to isolate the desired derivative $\frac{dy}{dx}$.

Example

Find $\frac{dy}{dx}$.

$$\sin(x+y) = 2x$$

$$\frac{d}{dx} \sin(x+y) = \frac{d}{dx}(2x)$$

$$\cos(x+y) \cdot \frac{d}{dx}(x+y) = 2$$

$$\cos(x+y)\left(1 + \frac{dy}{dx}\right) = 2$$

$$\cos(x+y) \cdot 1 + \cos(x+y)\frac{dy}{dx} = 2$$

$$\cos(x+y)\frac{dy}{dx} = 2 - \cos(x+y)$$

$$\boxed{\frac{dy}{dx} = \frac{2 - \cos(x+y)}{\cos(x+y)}}$$

$$\frac{dy}{dx} = \frac{2}{\cos(x+y)} - \frac{\cos(x+y)}{\cos(x+y)}$$

$$= 2\sec(x+y) - 1$$

Example

Find $\frac{dS}{dr}$.

$$e^{Sr} + S = r^2 + 2$$

S is dependent like y

r is independent like x

$$\frac{d}{dr}\left(e^{Sr} + S\right) = \frac{d}{dr}\left(r^2 + 2\right)$$

$$\frac{d}{dr} e^{Sr} + \frac{d}{dr} S = 2r + 0$$

$$e^{Sr} \frac{d}{dr}(Sr) + \frac{dS}{dr} = 2r$$

↑product

$$e^{Sr}\left(\frac{dS}{dr} r + S \cdot 1\right) + \frac{dS}{dr} = 2r$$

$$e^{sr} \cdot r \frac{ds}{dr} + s e^{sr} + \frac{ds}{dr} = 2r$$

$$re^{sr} \frac{ds}{dr} + \frac{ds}{dr} = 2r - se^{sr}$$

$$(re^{sr} + 1) \frac{ds}{dr} = 2r - se^{sr}$$

$$\boxed{\frac{ds}{dr} = \frac{2r - se^{sr}}{re^{sr} + 1}}$$