Sept. 26 Math 1190 sec. 52 Fall 2016

Section 3.1: The Chain Rule

The chain rule can be iterated to account for multiple compositions. For example, suppose f, g, h are appropriately differentiable, then

$$\frac{d}{dx}(f \circ g \circ h)(x) = \frac{d}{dx}f(g(h(x)) = f'(g(h(x))\,g'(h(x))\,h'(x)$$

Note that the outermost function is f, and its inner function is a composition $g(h(x))$.

So the derivative of the outer function evaluated at the inner is $f'(g(h(x))$ which is multiplied by the derivative of the inner function—**itself based on the chain rule**—$g'(h(x))h'(x)$.

If $h(x) = v$, $g(v) = u$, and $y = f(u)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}$$

Example

Evaluate the derivative $\dfrac{d}{dt}\tan^2\left(\dfrac{1}{3}t^3\right) = \dfrac{d}{dt}\left(\tan\left(\frac{1}{3}t^3\right)\right)^2$

Inner most $\quad v = h(t) = \frac{1}{3}t^3 \quad , \quad h'(t) = \frac{1}{3}(3t^2) = t^2$

middle $\quad u = g(v) = \tan v \quad , \quad g'(v) = \sec^2 v$

outer most $\quad f(u) = u^2 \quad , \quad f'(u) = 2u$

$$\frac{d}{dt}\tan^2\left(\frac{1}{3}t^3\right) = 2u \cdot \sec^2 v \cdot t^2$$

$$= 2\tan v \sec^2 v \cdot t^2$$

$u = \tan v$

$v = \frac{1}{3}t^3$

$$= 2\tan\left(\tfrac{1}{3}t^3\right)\sec^2\left(\tfrac{1}{3}t^3\right)\cdot t^2$$

$$= 2t^2 \tan\left(\tfrac{1}{3}t^3\right)\sec^2\left(\tfrac{1}{3}t^3\right)$$

Exponential of Base a

Let $a > 0$ with $a \neq 1$. By properties of logs and exponentials

$$a^x = e^{(\ln a)x}.$$

Note $(\ln a)x$ is the constant $\ln a$ times x.

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{(\ln a)x} = \underbrace{e^{(\ln a)x}}_{a^x} \cdot \ln a = a^x \ln a$$

$$= (\ln a)\, a^x$$

So $\frac{d}{dx} (\ln a) x$

$= (\ln a) \cdot 1$

$= \ln a$

Theorem: (Derivative of $y = a^x$) Let $a > 0$ and $a \neq 1$. Then

$$\frac{d}{dx} a^x = a^x \ln a$$

Example

$\frac{d}{dx} a^x = a^x \ln a$

Evaluate

(a) $\frac{d}{dx} 4^x$ $= 4^x \ln 4$

$f(g(x)) = 2^{\cos x} \Rightarrow g(x) = \cos x$

$f(u) = 2^u$

(b) $\frac{d}{dx} 2^{\cos x}$ $= \underbrace{2^{\cos x} (\ln 2)}_{2^u \ln 2} \cdot \underbrace{(-\sin x)}_{\frac{du}{dx}}$

$= -(\ln 2) \sin x \; 2^{\cos x}$

Inside $u = \cos x$

$\frac{du}{dx} = -\sin x$

Outside 2^u

$\frac{d}{du} 2^u = 2^u \ln 2$

Section 3.2: Implicit Differentiation; Derivatives of the Inverse Trigonometric Functions

The chain rule states that for a differentiable composition $f(g(x))$

$$\frac{d}{dx}\, f(g(x)) = f'(g(x))g'(x)$$

For $y = f(u)$ and $u = g(x)$

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$$

Example

Assume f is a differentiable function of x. Find an expression for the derivative:

$$\frac{d}{dx}\,(f(x))^2 = 2\,f(x)\cdot f'(x) = 2\,f(x)\,f'(x)$$

product 2 times f(x) times f'(x)

$$\frac{d}{dx}\tan(f(x)) = \sec^2(f(x))\cdot f'(x)$$

Secant squared of f(x) times f'(x)

Example

Suppose we know that $y = f(x)$ for some differentiable function (but we don't know exactly what f is). Find an expression for the derivative.

$$\frac{d}{dx} y^3 = 3y^2 \cdot \frac{dy}{dx} = 3y^2 \frac{dy}{dx}$$

y is an inside function

u^3 is an outside function

3 times y^2 times $\frac{dy}{dx}$

$$\frac{d}{dx} x^2y^2 = \left(\frac{d}{dx} x^2\right) y^2 + x^2 \left(\frac{d}{dx} y^2\right) = 2xy^2 + x^2\left(2y \cdot \frac{dy}{dx}\right)$$

$$= 2xy^2 + 2x^2y \frac{dy}{dx}$$

x^2 times y^2 is a product

product rule

product rule

$$\frac{d}{dx} fg = f'g + fg'$$

Implicitly defined functions

A relation—an equation involving two variables x and y—such as

$$x^2 + y^2 = 16 \quad \text{or} \quad (x^2 + y^2)^3 = x^2$$

implies that y is defined to be one or more functions of x.

e.g $x^2+y^2=16 \Rightarrow y^2 = 16-x^2$

$\Rightarrow y = \sqrt{16-x^2}$ or $y = -\sqrt{16-x^2}$

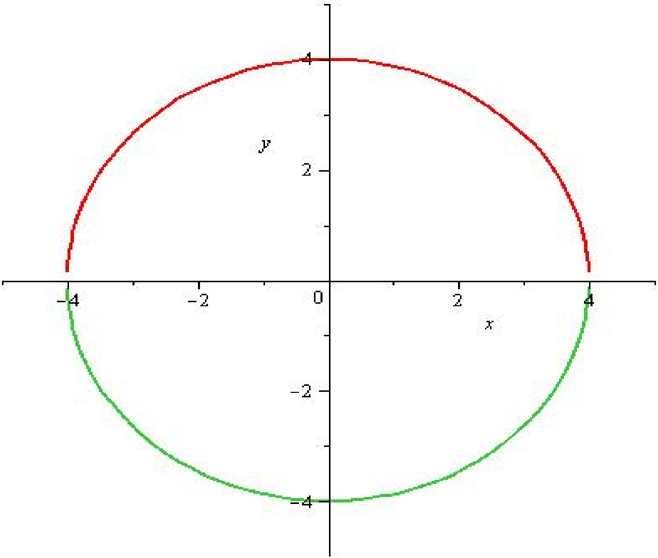

Figure: $x^2 + y^2 = 16$

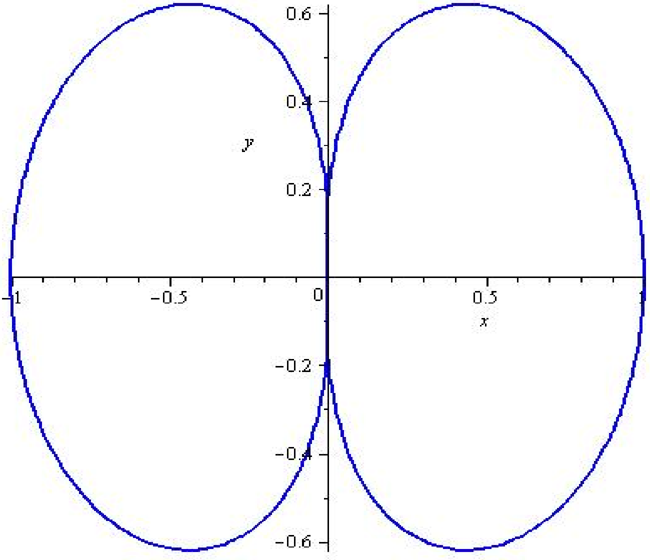

Figure: $(x^2 + y^2)^3 = x^2$

Explicit -vs- Implicit

A function is defined **explicitly** when given in the form

$$y = f(x).$$

e.g. $y = x^2$ or $y = e^{\tan x}$ or $y = \ln(\cos x)$

A function is defined *implicitly* when it is given as a relation

$$F(x, y) = C,$$

for constant C. e.g. $x^2 + y^2 = 16$ or $(x^2+y^2)^3 = x^2$

or $\cos(x^2 y^2) = 2x$

Implicit Differentiation

Since $x^2 + y^2 = 16$ *implies* that y is a function of x, we can consider it's derivative.

$$\text{Find } \frac{dy}{dx} \text{ given } \quad x^2 + y^2 = 16.$$

Take the derivative $\frac{d}{dx}$, with respect to x, of both sides of the relation.

$$\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}(16)$$

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = 0$$

$$2x + 2y \cdot \frac{dy}{dx} = 0$$

Isolate $\frac{dy}{dx}$ using algebra.

put $\frac{dy}{dx}$ terms on the left, all others on the right

$$2y\frac{dy}{dx} = -2x$$

Divide by the coefficient $2y$

$$\frac{dy}{dx} = \frac{-2x}{2y} \Rightarrow \boxed{\frac{dy}{dx} = \frac{-x}{y}}$$

Show that the same result is obtained knowing

$$y = \sqrt{16 - x^2} \quad \text{or} \quad y = -\sqrt{16 - x^2}.$$

Square root derivative rule: $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$

$$\frac{d}{dx} y = \frac{d}{dx}\sqrt{16 - x^2} = \frac{1}{2\sqrt{16-x^2}} \cdot (0 - 2x)$$

inside $16 - x^2$

outside $\sqrt{u}$

$$= \frac{-2x}{2\sqrt{16 - x^2}}$$

$$= \frac{-x}{\sqrt{16 - x^2}} = \frac{-x}{y}$$

Since $y = \sqrt{16 - x^2}$

If $y = -\sqrt{16-x^2}$

$$\frac{d}{dx} y = \frac{d}{dx}\left(-\sqrt{16-x^2}\right) = -\frac{d}{dx}\sqrt{16-x^2}$$

$$= -\left(\frac{-x}{\sqrt{16-x^2}}\right)$$

$$= \frac{-x}{-\sqrt{16-x^2}} = \frac{-x}{y}$$ since $y = -\sqrt{16-x^2}$

Example

$$\text{Find } \frac{dy}{dx} \text{ given } \quad x^2 - 3xy + y^2 = y.$$

Find $\frac{dy}{dx}$ of both sides.

$$\frac{d}{dx}\left(x^2 - 3xy + y^2\right) = \frac{d}{dx}\, y$$

$$\frac{d}{dx}x^2 - 3\underbrace{\frac{d}{dx}(xy)}_{\text{product}} + \frac{d}{dx}y^2 = \frac{dy}{dx}$$

$$2x - 3\left(1 \cdot y + x\frac{dy}{dx}\right) + 2y \cdot \frac{dy}{dx} = \frac{dy}{dx}$$

Now isolate $\frac{dy}{dx}$ w/ algebra

get $\frac{dy}{dx}$ terms on the left
all others on the right

$$2x - 3y - 3x\frac{dy}{dx} + 2y\frac{dy}{dx} = \frac{dy}{dx}$$

$$-3x\frac{dy}{dx} + 2y\frac{dy}{dx} - \frac{dy}{dx} = 3y - 2x$$

$$(-3x + 2y - 1)\frac{dy}{dx} = 3y - 2x$$

$$\frac{dy}{dx} = \frac{3y - 2x}{-3x + 2y - 1}$$

$$\boxed{\frac{dy}{dx} = \frac{3y - 2x}{2y - 3x - 1}}$$

Finding a Derivative Using Implicit Differentiation:

- Take the derivative of both sides of an equation with respect to the independent variable.
- Use all necessary rules for differenting powers, products, quotients, trig functions, exponentials, compositions, etc.
- Remember the chain rule for each term involving the dependent variable (e.g. mult. by $\frac{dy}{dx}$ as required).
- Use necessary algebra to isolate the desired derivative $\frac{dy}{dx}$.

Example

Find $\frac{dy}{dx}$.

$$\sin(x+y) = 2x$$

$$\frac{d}{dx}\sin(x+y) = \frac{d}{dx}(2x)$$

$$\cos(x+y)\frac{d}{dx}(x+y) = 2$$

$$\cos(x+y)\left[1 + \frac{dy}{dx}\right] = 2$$

$$\cos(x+y)\cdot 1 + \cos(x+y)\frac{dy}{dx} = 2$$

$$\cos(x+y)\frac{dy}{dx} = 2 - \cos(x+y)$$

$$\boxed{\frac{dy}{dx} = \frac{2-\cos(x+y)}{\cos(x+y)}}$$

$$\frac{dy}{dx} = \frac{2}{\cos(x+y)} - \frac{\cos(x+y)}{\cos(x+y)}$$

$$= 2\sec(x+y) - 1$$

Example

Find $\frac{dS}{dr}$.

$$e^{Sr} + S = r^2 + 2$$

S is dependent like y

r is independent like x

$$\frac{d}{dr}\left(e^{Sr} + S\right) = \frac{d}{dr}\left(r^2 + 2\right)$$

$$\frac{d}{dr} e^{Sr} + \frac{d}{dr} S = 2r + 0$$

$$e^{Sr} \frac{d}{dr}(Sr) + \frac{dS}{dr} = 2r$$

$$e^{Sr}\left(\frac{dS}{dr} r + S \cdot 1\right) + \frac{dS}{dr} = 2r$$

$$re^{Sr}\frac{dS}{dr} + Se^{Sr} + \frac{dS}{dr} = 2r$$

$$re^{Sr}\frac{dS}{dr} + \frac{dS}{dr} = 2r - Se^{Sr}$$

$$(re^{Sr}+1)\frac{dS}{dr} = 2r - Se^{Sr}$$

$$\boxed{\frac{dS}{dr} = \frac{2r - Se^{Sr}}{re^{Sr}+1}}$$