Sept 26 Math 2306 sec. 53 Fall 2018

Section 8: Homogeneous Equations with Constant Coefficients

We consider a second order, linear, homogeneous equation with constant coefficients

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0.$$

Suppose $y=e^{mx}$

The equation has **characteristic** (also called **auxiliary**) equation

$$am^2+bm+c=0$$

The solutions to the ODE come in three different cases based on the roots of the characteristic equation.

Cases I & II: $ay'' + by' + cy = 0$

Case I: $b^2 - 4ac > 0$. The characteristic equation has two distinct real roots m_1 and m_2. Then a fundamental solution set is

$$y_1 = e^{m_1 x} \quad \text{and} \quad y_2 = e^{m_2 x}$$

and the general solution is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

Case II: $b^2 - 4ac = 0$. The characteristic equation has one repeated real root m. Then a fundamental solution set is

$$y_1 = e^{mx} \quad \text{and} \quad y_2 = xe^{mx}$$

and the general solution is

$$y = c_1 e^{mx} + c_2 x e^{mx}.$$

Case III: Complex conjugate roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac < 0$$

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x)), \quad \text{where the roots}$$

$$m = \alpha \pm i\beta, \quad \alpha = \frac{-b}{2a} \quad \text{and} \quad \beta = \frac{\sqrt{4ac - b^2}}{2a}$$

The solutions can be written as

$$Y_1 = e^{(\alpha + i\beta)x} = e^{\alpha x} e^{i\beta x}, \quad \text{and} \quad Y_2 = e^{(\alpha - i\beta)x} = e^{\alpha x} e^{-i\beta x}.$$

Deriving the solutions Case III

Recall Euler's Formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$y_1 = e^{\alpha x} e^{i\beta x} = e^{\alpha x}\left(\cos(\beta x) + i\sin(\beta x)\right)$$

$$= e^{\alpha x}\cos(\beta x) + i\, e^{\alpha x}\sin(\beta x)$$

$$y_2 = e^{\alpha x} e^{-i\beta x} = e^{\alpha x}\left(\cos(\beta x) - i\sin(\beta x)\right)$$

$$= e^{\alpha x}\cos(\beta x) - i\, e^{\alpha x}\sin(\beta x)$$

Let

$$y_1 = \frac{1}{2}(\psi_1 + \psi_2) = \frac{1}{2}\left(2e^{\alpha x}\cos(\beta x)\right) = e^{\alpha x}\cos(\beta x)$$

$$y_2 = \frac{1}{2i}(\psi_1 - \psi_2) = \frac{1}{2i}\left(2i\,e^{\alpha x}\sin(\beta x)\right) = e^{\alpha x}\sin(\beta x)$$

So our fundamental solution set will be

$$y_1 = e^{\alpha x}\cos(\beta x), \quad y_2 = e^{\alpha x}\sin(\beta x)$$

Example

Solve the ODE

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 6x = 0$$

Characteristic eqn. $m^2 + 4m + 6 = 0$

Completing the square

$$m^2 + 4m + 4 - 4 + 6 = 0$$

$$(m+2)^2 + 2 = 0$$

$$(m+2)^2 = -2$$

$$m + 2 = \pm\sqrt{-2} = \pm\sqrt{2}\, i$$

$m = -2 \pm \sqrt{2}\, i$ $\quad\quad$ $m = \alpha \pm i\beta$

So $\alpha = -2$ and $\beta = \sqrt{2}$

$x_1 = e^{-2t}\cos(\sqrt{2}\,t)$ and $x_2 = e^{-2t}\sin(\sqrt{2}\,t)$

The general solution is

$$x = c_1 e^{-2t}\cos(\sqrt{2}\,t) + c_2 e^{-2t}\sin(\sqrt{2}\,t)$$

Higer Order Linear Constant Coefficient ODEs

- The same approach applies. For an n^{th} order equation, we obtain an n^{th} degree polynomial.
- Complex roots must appear in conjugate pairs (due to real coefficients) giving a pair of solutions $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$.
- If a root m is repeated k times, we get k linearly independent solutions

  $$e^{mx}, \quad xe^{mx}, \quad x^2e^{mx}, \quad \ldots, \quad x^{k-1}e^{mx}$$

  or in conjugate pairs cases $2k$ solutions

  $$e^{\alpha x}\cos(\beta x),\ e^{\alpha x}\sin(\beta x), \quad xe^{\alpha x}\cos(\beta x),\ xe^{\alpha x}\sin(\beta x), \ldots,$$
  $$x^{k-1}e^{\alpha x}\cos(\beta x),\ x^{k-1}e^{\alpha x}\sin(\beta x)$$

- It may require a computer algebra system to find the roots for a high degree polynomial.

Example

Solve the ODE

$$y''' - 4y' = 0$$

Let $y = e^{mx}$ $y' = me^{mx}$ $y'' = m^2 e^{mx}$ $y''' = m^3 e^{mx}$

$m^3 e^{mx} - 4me^{mx} = 0$ Char. eqn.

$e^{mx}(m^3 - 4m) = 0 \Rightarrow m^3 - 4m = 0$

$m(m^2-4) = 0 \Rightarrow m(m-2)(m+2) = 0$

$m_1 = 0,\ m_2 = 2,\ m_3 = -2$

$y_1 = e^{0x} = 1$

$y_2 = e^{2x}$

$y_3 = e^{-2x}$

The general solution is

$$y = C_1 + C_2 e^{2x} + C_3 e^{-2x}$$