Section 8: Homogeneous Equations with Constant Coefficients

We’re considering second order, linear, homogeneous equations with constant coefficients

\[ a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0. \]

We sought solutions of the form \( y = e^{mx} \) and derived an associated characteristic (a.k.a. auxiliary equation) for \( m \)

\[ am^2 + bm + c = 0. \]

Three cases have to be considered regarding the roots of this equation.
Case I: Two distinct real roots

From \( am^2 + bm + c = 0 \), if \( b^2 - 4ac > 0 \), there are two different real number roots

\[
m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

giving two linearly independent solutions

\[
y_1 = e^{m_1x} \quad \text{and} \quad y_2 = e^{m_2x}
\]

for a general solution

\[
y = c_1 e^{m_1x} + c_2 e^{m_2x}.
\]
Case II: One repeated real root

\[ ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac = 0 \]

\[ y = c_1 e^{mx} + c_2 x e^{mx} \quad \text{where} \quad m = \frac{-b}{2a} \]

Use reduction of order to show that if \( y_1 = e^{\frac{-bx}{2a}} \), then \( y_2 = xe^{\frac{-bx}{2a}} \).

\[ y_2 = u(x)y_1, \quad \text{where} \quad u = \int \frac{e^{-\int p(x)dx}}{(y_1)^2} \, dx \]

Standard form:

\[ y'' + \frac{b}{a} y' + \frac{c}{a} y = 0 \quad p(x) = \frac{b}{a} \quad \text{constant} \]

\[ -\int p(x) \, dx = -\int \frac{b}{a} \, dx = -\frac{b}{a} x \]
\[ u = \int e^{\frac{-b}{2a}x} \, dx = \int \frac{e^{-\frac{b}{2a}x}}{(e^{\frac{-b}{2a}x})^2} \, dx \]

\[ = \int \frac{e^{\frac{b}{2a}x}}{e^{-\frac{b}{2a}x}} \, dx = \int \, dx = x \]

\[ y_2 = uy_1 = x e^{\frac{-b}{2a}x} \]
Example

Solve the ODE

\[ 4y'' - 4y' + y = 0 \]

Characteristic Eqn: \[ 4m^2 - 4m + 1 = 0 \]
\[ (2m - 1)^2 = 0 \]
\[ m = \frac{1}{2} \text{ repeated} \]

\[ y_1 = e^{\frac{1}{2}x}, \quad y_2 = xe^{\frac{1}{2}x} \]

The general solution is \( y = c_1 e^{\frac{1}{2}x} + c_2 xe^{\frac{1}{2}x} \)
Case III: Complex conjugate roots

\[ ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac < 0 \]
\[ y = e^{\alpha x} (c_1 \cos(\beta x) + c_2 \sin(\beta x)), \quad \text{where the roots} \]
\[ m = \alpha \pm i\beta, \quad \alpha = \frac{-b}{2a} \quad \text{and} \quad \beta = \frac{\sqrt{4ac - b^2}}{2a} \]

The solutions can be written as
\[ Y_1 = e^{(\alpha + i\beta)x} = e^{\alpha x} e^{i\beta x}, \quad \text{and} \quad Y_2 = e^{(\alpha - i\beta)x} = e^{\alpha x} e^{-i\beta x}. \]
Deriving the solutions Case III

Recall Euler’s Formula:

\[ e^{i\theta} = \cos \theta + i \sin \theta \]

\( \gamma_1 = e^{\alpha x} e^{i\beta x} = e^{\alpha x} (\cos(\beta x) + i \sin(\beta x)) \)

\[ = e^{\alpha x} \cos(\beta x) + i e^{\alpha x} \sin(\beta x) \]

\( \gamma_2 = e^{\alpha x} e^{-i\beta x} = e^{\alpha x} (\cos(\beta x) - i \sin(\beta x)) \)

\[ = e^{\alpha x} \cos(\beta x) - i e^{\alpha x} \sin(\beta x) \]
Define

\[ y_1 = \frac{1}{2}(y_1 + y_2) = \frac{1}{2} \left( 2e^{\alpha x} \cos(\beta x) + 0 \right) \]

\[ y_1 = e^{\alpha x} \cos(\beta x) \]

\[ y_2 = \frac{1}{2i}(y_1 - y_2) = \frac{1}{2i} \left( 0 + 2i e^{\alpha x} \sin(\beta x) \right) \]

\[ y_2 = e^{\alpha x} \sin(\beta x) \]

A fundamental solution set is

\[ \{ e^{\alpha x} \cos(\beta x), e^{\alpha x} \sin(\beta x) \} \]
Example

Solve the ODE

\[ \frac{d^2 x}{dt^2} + 4 \frac{dx}{dt} + 6x = 0 \]

Characteristic eqn: \[ m^2 + 4m + 6 = 0 \]

\[ m_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 1 \cdot 6}}{2} \]

\[ = \frac{-4 \pm \sqrt{-8}}{2} \]

\[ = -2 \pm \sqrt{2}i \]

\[ X_1 = e^{-2t} \cos(\sqrt{2}t), \quad X_2 = e^{-2t} \sin(\sqrt{2}t) \]

The general solution is

\[ X = c_1 e^{-2t} \cos(\sqrt{2}t) + c_2 e^{-2t} \sin(\sqrt{2}t) \]
Higher Order Linear Constant Coefficient ODEs

- The same approach applies. For an \( n^{\text{th}} \) order equation, we obtain an \( n^{\text{th}} \) degree polynomial.

- Complex roots must appear in conjugate pairs (due to real coefficients) giving a pair of solutions \( e^{\alpha x} \cos(\beta x) \) and \( e^{\alpha x} \sin(\beta x) \).

- If a root \( m \) is repeated \( k \) times, we get \( k \) linearly independent solutions

\[
e^{mx}, \quad xe^{mx}, \quad x^2 e^{mx}, \quad \ldots, \quad x^{k-1} e^{mx}
\]

or in conjugate pairs cases \( 2k \) solutions

\[
e^{\alpha x} \cos(\beta x), \quad e^{\alpha x} \sin(\beta x), \quad xe^{\alpha x} \cos(\beta x), \quad xe^{\alpha x} \sin(\beta x), \quad \ldots,
\]

\[
x^{k-1} e^{\alpha x} \cos(\beta x), \quad x^{k-1} e^{\alpha x} \sin(\beta x)
\]

- It may require a computer algebra system to find the roots for a high degree polynomial.
Example

Solve the ODE

\[ y''' - 4y' = 0 \]

3rd order: Characteristic Eqn

\[ m^3 - 4m = 0 \]
\[ m(m^2 - 4) = 0 \rightarrow m(m-2)(m+2) = 0 \]
\[ m_1 = 0, \ m_2 = 2, \ m_3 = -2 \]

\[ y_1 = e^0 = 1, \ y_2 = e^{2x}, \ y_3 = e^{-2x} \]

The general solution is

\[ y = c_1 + c_2 e^{2x} + c_3 e^{-2x} \]
Example

Solve the ODE

\[ y''' - 3y'' + 3y' - y = 0 \]

Characteristic Eqn

\[ m^3 - 3m^2 + 3m - 1 = 0 \]
\[ (m-1)^3 = 0 \]

\[ m = 1, \text{ triple root} \]

\[ y_1 = e^x, \quad y_2 = xe^x, \quad y_3 = x^2e^x \]

The general solution is

\[ y = c_1e^x + c_2xe^x + c_3x^2e^x \]
Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

\[ a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x) \]

where \( g \) comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

Recall \( y = y_c + y_p \), so we’ll have to find both the complementary and the particular solutions!