September 26 Math 2306 sec. 56 Fall 2017

Section 8: Homogeneous Equations with Constant Coefficients

We're considering second order, linear, homogeneous equations with constant coefficients

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0.$$

We sought solutions of the form $y = e^{mx}$ and derived an associated characteristic (a.k.a. auxiliary equation) for m

$$am^2+bm+c=0.$$

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Three cases have to be considered regarding the roots of this equation.

Case I: Two distinct real roots

From $am^2 + bm + c = 0$, if $b^2 - 4ac > 0$, there are two different real number roots

$$m_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

giving two linearly independent solutions

$$y_1 = e^{m_1 x}$$
 and $y_2 = e^{m_2 x}$

for a general solution

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

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Case II: One repeated real root

$$ay'' + by' + cy = 0$$
, where $b^2 - 4ac = 0$
 $y = c_1e^{mx} + c_2xe^{mx}$ where $m = \frac{-b}{2a}$

Use reduction of order to show that if $y_1 = e^{\frac{-bx}{2a}}$, then $y_2 = xe^{\frac{-bx}{2a}}$.

$$y_{z} = u(x)y_{1}, \quad \text{where} \quad u = \int \frac{e^{-y_{1}(x)x_{2}}}{(y_{1})^{2}} dx$$

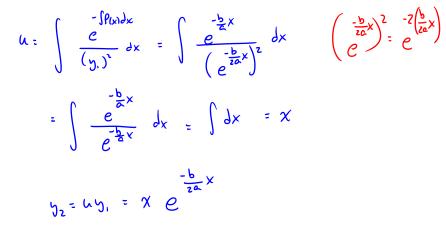
Standard form:
$$y'' + \frac{b}{a}y' + \frac{c}{a}y = 0 \quad P(x) = \frac{b}{a} \quad (\text{oretherm}^{t})^{t}$$
$$-\int P(x) dx = -\int \frac{b}{a} dx = -\frac{b}{a}x$$

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Example

Solve the ODE

$$4y'' - 4y' + y = 0$$
Charachistic Eqn: $4m^2 - 4m + 1 = 0$

$$(2m - 1)^2 = 0$$

$$m = \frac{1}{2} \quad \text{repeated}$$

$$y_1 = e^{-1}, \quad y_2 = xe^{-1}$$
The general colution is $y = c_1 e^{-1} + c_2 x e^{-1}$

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Case III: Complex conjugate roots

$$ay'' + by' + cy = 0$$
, where $b^2 - 4ac < 0$
 $y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$, where the roots $m = \alpha \pm i\beta$, $\alpha = \frac{-b}{2a}$ and $\beta = \frac{\sqrt{4ac - b^2}}{2a}$

The solutions can be written as

$$Y_1 = e^{(\alpha + i\beta)x} = e^{\alpha x} e^{i\beta x}$$
, and $Y_2 = e^{(\alpha - i\beta)x} = e^{\alpha x} e^{-i\beta x}$.

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Deriving the solutions Case III

Recall Euler's Formula:

 $e^{i\theta} = \cos\theta + i\sin\theta$ Y = e e = e (Cos(px) + i Sin(px)) = edx Cos (Bx) + i edx Sin (Bx) $Y_2 = e^{\alpha x} e^{-i\beta x} = e^{\alpha x} \left(C_{os}(\beta x) - i S_{in}(\beta x) \right)$ = e^{dx} (os (px) - i e^{dx} Sim(px)

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Pulline

$$y_{i} = \frac{1}{2} \left(Y_{i} + Y_{2} \right) = \frac{1}{2} \left(2e^{AX} \left(\cos(\beta x) + 0 \right) \right)$$

$$y_{i} = e^{AX} \cos(\beta x)$$

$$y_{i} = \frac{1}{2i} \left((Y_{i} - Y_{2}) \right) = \frac{1}{2i} \left(0 + 2i e^{AX} \sin(\beta x) \right)$$

$$y_{i} = e^{AX} \sin(\beta x)$$

$$A \quad f_{indocated} \quad solution \quad set \quad is$$

$$\left\{ e^{AX} \left(\cos(\beta x) \right), e^{AX} \sin(\beta x) \right\}.$$

$$(1 + e^{AX} + e^{AX}) = \frac{1}{2i} \left(e^{AX} \sin(\beta x) \right)$$

Example

Solve the ODE $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 6x = 0$ $m^{2} + 4m + 6 = 0$ M.,= Characteristic egn: 9=-2, B=JZ $X_1 = e^{2t} Cos(52t), \quad X_2 = e^{2t} Sin(52t)$ -2±50 Served Solution is $X = c_1 e^{-2t} c_0 r(5t) + c_2 e^{-2t} Sim(5t)$ September 26, 2017 9/45

Higer Order Linear Constant Coefficient ODEs

- The same approach applies. For an nth order equation, we obtain an nth degree polynomial.
- Complex roots must appear in conjugate pairs (due to real coefficients) giving a pair of solutions e^{αx} cos(βx) and e^{αx} sin(βx).
- If a root m is repeated k times, we get k linearly independent solutions

$$e^{mx}$$
, xe^{mx} , x^2e^{mx} , ..., $x^{k-1}e^{mx}$

or in conjugate pairs cases 2k solutions

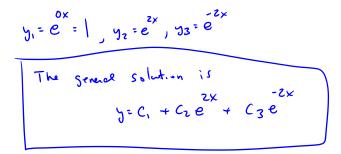
 $e^{\alpha x}\cos(\beta x), e^{\alpha x}\sin(\beta x), xe^{\alpha x}\cos(\beta x), xe^{\alpha x}\sin(\beta x), \dots,$

 $x^{k-1}e^{\alpha x}\cos(\beta x), x^{k-1}e^{\alpha x}\sin(\beta x)$

It may require a computer algebra system to find the roots for a high degree polynomial.

Example

Solve the ODE 3^{rd} orden: Characteristic fin y'''-4y'=0 $m_{(m^{2}-4)}^{3}=0 \implies m(m-2)(m+2)=0$ $m_{(m^{2}-4)}^{3}=0 \implies m(m-2)(m+2)=0$ $m_{(m^{2}-4)}^{3}=0 \implies m(m-2)(m+2)=0$



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Example

Solve the ODE

$$y'''-3y''+3y'-y=0$$

$$y'''-3y''+3y'-y=0$$

$$y_{1}=e^{x}, y_{2}=xe^{y}, y_{3}=x^{2}e^{x}$$

$$y_{1}=e^{x}, y_{2}=xe^{y}, y_{3}=x^{2}e^{x}$$

$$M=1, triple root$$

$$The general solution is$$

$$y_{1}=c_{1}e^{x}+c_{2}xe^{z}+c_{3}xe^{z}$$

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Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

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