September 26 Math 2306 sec. 57 Fall 2017

Section 8: Homogeneous Equations with Constant Coefficients

We're considering second order, linear, homogeneous equations with constant coefficients

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0.$$

We sought solutions of the form $y=e^{mx}$ and derived an associated characteristic (a.k.a. auxiliary equation) for m

$$am^2+bm+c=0.$$

Three cases have to be considered regarding the roots of this equation.

Case I: Two distinct real roots

From $am^2 + bm + c = 0$, if $b^2 - 4ac > 0$, there are two different real number roots

$$m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

giving two linearly independent solutions

$$y_1 = e^{m_1 x} \quad \text{and} \quad y_2 = e^{m_2 x}$$

for a general solution

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

Case II: One repeated real root

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac = 0$$

$$y = c_1 e^{mx} + c_2 x e^{mx} \quad \text{where} \quad m = \frac{-b}{2a}$$

Use reduction of order to show that if $y_1 = e^{\frac{-bx}{2a}}$, then $y_2 = xe^{\frac{-bx}{2a}}$.

$y_2 = u y_1$ where $u = \int \frac{e^{-\int P(x)\,dx}}{(y_1)^2}\,dx$

Standard form: $y'' + \frac{b}{a} y' + \frac{c}{a} y = 0 \implies P(x) = \frac{b}{a}$ constant

$-\int P(x)\,dx = -\int \frac{b}{a}\,dx = -\frac{b}{a} x$ so $e^{-\int P(x)\,dx} = e^{-\frac{b}{a}x}$

$$u = \int \frac{e^{-\int P(x)\,dx}}{(y_1)^2}\,dx = \int \frac{e^{-\frac{b}{a}x}}{\left(e^{-\frac{b}{2a}x}\right)^2}\,dx \qquad \left(e^{-\frac{b}{2a}x}\right)^2 = e^{2\left(-\frac{b}{2a}x\right)}$$

$$= \int \frac{e^{-\frac{b}{a}x}}{e^{-\frac{b}{a}x}}\,dx = \int dx = x$$

$$y_2 = u y_1 = x\,e^{-\frac{b}{2a}x}$$

Example

Solve the ODE

$$4y'' - 4y' + y = 0$$

Characteristic eqn: $4m^2 - 4m + 1 = 0 \Rightarrow (2m-1)^2 = 0$

$m = \frac{1}{2}$ repeated

$y_1 = e^{\frac{1}{2}x}, \quad y_2 = xe^{\frac{1}{2}x}$

The general solution is

$$\boxed{y = c_1 e^{\frac{1}{2}x} + c_2 x e^{\frac{1}{2}x}}$$

Case III: Complex conjugate roots

$$ay'' + by' + cy = 0, \quad \text{where} \quad b^2 - 4ac < 0$$

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x)), \quad \text{where the roots}$$

$$m = \alpha \pm i\beta, \quad \alpha = \frac{-b}{2a} \quad \text{and} \quad \beta = \frac{\sqrt{4ac - b^2}}{2a}$$

The solutions can be written as

$$Y_1 = e^{(\alpha + i\beta)x} = e^{\alpha x} e^{i\beta x}, \quad \text{and} \quad Y_2 = e^{(\alpha - i\beta)x} = e^{\alpha x} e^{-i\beta x}.$$

Deriving the solutions Case III

Recall Euler's Formula:

βx is θ

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$y_1 = e^{\alpha x} e^{i\beta x} = e^{\alpha x}\left(\cos(\beta x) + i\sin(\beta x)\right)$$

$$= e^{\alpha x}\cos(\beta x) + i\, e^{\alpha x}\sin(\beta x)$$

$$y_2 = e^{\alpha x} e^{-i\beta x} = e^{\alpha x}\left(\cos(\beta x) - i\sin(\beta x)\right)$$

$$= e^{\alpha x}\cos(\beta x) - i\, e^{\alpha x}\sin(\beta x)$$

Set

$$y_1 = \frac{1}{2}(Y_1 + Y_2) = \frac{1}{2}\left(2e^{\alpha x}\cos(\beta x) + 0\right)$$

$$= e^{\alpha x}\cos(\beta x)$$

and

$$y_2 = \frac{1}{2i}(Y_1 - Y_2) = \frac{1}{2i}\left(0 + 2i\, e^{\alpha x}\sin(\beta x)\right)$$

$$= e^{\alpha x}\sin(\beta x)$$

These are linearly independent.

The gen. soln is

$$y = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x)$$

Example

Solve the ODE

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 6x = 0$$

Char. eqn $m^2 + 4m + 6 = 0$

$$m = \frac{-4 \pm \sqrt{4^2 - 4\cdot1\cdot6}}{2\cdot1} = \frac{-4 \pm \sqrt{-8}}{2} = \frac{-4 \pm 2\sqrt{2}\,i}{2} = -2 \pm \sqrt{2}\,i$$

$\alpha = -2$

$\beta = \sqrt{2}$

$x_1 = e^{-2t}\cos(\sqrt{2}\,t)$

$x_2 = e^{-2t}\sin(\sqrt{2}\,t)$

The general soln is

$$\boxed{x = c_1 e^{-2t}\cos(\sqrt{2}\,t) + c_2 e^{-2t}\sin(\sqrt{2}\,t)}$$

Higer Order Linear Constant Coefficient ODEs

- The same approach applies. For an n^{th} order equation, we obtain an n^{th} degree polynomial.
- Complex roots must appear in conjugate pairs (due to real coefficients) giving a pair of solutions $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$.
- If a root m is repeated k times, we get k linearly independent solutions

$$e^{mx}, \quad xe^{mx}, \quad x^2 e^{mx}, \quad \ldots, \quad x^{k-1}e^{mx}$$

  or in conjugate pairs cases $2k$ solutions

$$e^{\alpha x}\cos(\beta x),\ e^{\alpha x}\sin(\beta x), \quad xe^{\alpha x}\cos(\beta x),\ xe^{\alpha x}\sin(\beta x), \ldots,$$

$$x^{k-1}e^{\alpha x}\cos(\beta x),\ x^{k-1}e^{\alpha x}\sin(\beta x)$$

- It may require a computer algebra system to find the roots for a high degree polynomial.

Example

Solve the ODE

$y''' - 4y' = 0$

Characteristic eqn $m^3 - 4m = 0$

$m(m^2-4) = 0$

$m(m-2)(m+2) = 0$

$m_1 = 0, \ m_2 = 2, \ m_3 = -2.$

$y_1 = e^{0x} = 1, \ y_2 = e^{2x}, \ y_3 = e^{-2x}$

The general solution is

$$y = c_1 + c_2 e^{2x} + c_3 e^{-2x}.$$

Example

Solve the ODE

$$y''' - 3y'' + 3y' - y = 0$$

Charc. eqn $m^3 - 3m^2 + 3m - 1 = 0$

$(m-1)^3 = 0$

$m = 1$, triple root

$y_1 = e^x$, $y_2 = xe^x$, $y_3 = x^2e^x$

General solution

$$y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x$$

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

Motivating Example

Find a particular solution of the ODE

$$y'' - 4y' + 4y = 8x + 1$$

We'll guess that y_p matches the "form" of $g(x) = 8x+1$. g is a line, so we'll guess that $y_p = Ax+B$ for some constants A and B.

$y_p' = A$, $y_p'' = 0$

so $y_p'' - 4y_p' + 4y_p = 0 - 4(A) + 4(Ax+B) = 8x+1$

A and B should satisfy

$$-4A + 4Ax + 4B = 8x + 1$$

$$4Ax + (-4A + 4B) = 8x + 1$$

Matching coefficients

$$4A = 8 \quad \text{and} \quad -4A + 4B = 1$$

$$A = 2 \implies 4B = 1 + 4A = 1 + 4 \cdot 2 = 9$$

$$B = \frac{9}{4}$$

$$\boxed{\text{So it works with } y_p = 2x + \frac{9}{4}}$$

Check $\quad y_p = 2x + \frac{9}{4}, \quad y_p' = 2, \quad y_p'' = 0$

$$y_p'' - 4y_p' + 4y_p = 0 - 4(2) + 4\left(2x + \frac{9}{4}\right)$$

$$= -8 + 8x + 9$$

$$= 8x + 1$$