September 26 Math 2306 sec. 57 Fall 2017

Section 8: Homogeneous Equations with Constant Coefficients

We're considering second order, linear, homogeneous equations with constant coefficients

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0.$$

We sought solutions of the form $y = e^{mx}$ and derived an associated characteristic (a.k.a. auxiliary equation) for m

$$am^2+bm+c=0.$$

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Three cases have to be considered regarding the roots of this equation.

Case I: Two distinct real roots

From $am^2 + bm + c = 0$, if $b^2 - 4ac > 0$, there are two different real number roots

$$m_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

giving two linearly independent solutions

$$y_1 = e^{m_1 x}$$
 and $y_2 = e^{m_2 x}$

for a general solution

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

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Case II: One repeated real root

$$ay'' + by' + cy = 0$$
, where $b^2 - 4ac = 0$
 $y = c_1e^{mx} + c_2xe^{mx}$ where $m = \frac{-b}{2a}$

Use reduction of order to show that if $y_1 = e^{\frac{-bx}{2a}}$, then $y_2 = xe^{\frac{-bx}{2a}}$.

$$y_{2} = U y_{1} \text{ when } U = \int \frac{e^{-\int P(x) dx}}{(y_{1})^{2}} dx$$

Standard form: $y'' + \frac{b}{a} y' + \frac{c}{a} y = 0 \Rightarrow P(x) = \frac{b}{a} \operatorname{constant}$
 $-\int P(x) dx = -\int \frac{b}{a} dx = -\frac{b}{a} \times s_{1} = \int P(x) dx = -\frac{b}{a} \times s_{2}$

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$$u = \int \frac{e}{(y_1)^2} \frac{dx}{dx} = \int \frac{e^{-\frac{b}{2}x}}{\left(e^{-\frac{b}{2}x}\right)^2} \frac{dx}{dx} \qquad \left(e^{-\frac{b}{2}x}\right)^2 = e^{(\frac{b}{2}x)}$$
$$= \int \frac{e^{-\frac{b}{2}x}}{e^{-\frac{b}{2}x}} \frac{dx}{dx} = \int \frac{dx}{dx} = \chi$$
$$y_2 = uy_1 = \chi e^{-\frac{b}{2}x}$$

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Example

Solve the ODE 4v'' - 4v' + v = 0Charadeirthe eqn: $4m^2 - 4m + 1 = 0 \Rightarrow (2m - 1)^2 = 0$ M= 12 repeated y.= e[±], y.= xe[±]x Several solution is $y = c_1 e^{\frac{1}{2}x} + c_2 x e^{\frac{1}{2}x}$

Case III: Complex conjugate roots

$$ay'' + by' + cy = 0$$
, where $b^2 - 4ac < 0$
 $y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$, where the roots $m = \alpha \pm i\beta$, $\alpha = \frac{-b}{2a}$ and $\beta = \frac{\sqrt{4ac - b^2}}{2a}$

The solutions can be written as

$$Y_1 = e^{(\alpha + i\beta)x} = e^{\alpha x} e^{i\beta x}$$
, and $Y_2 = e^{(\alpha - i\beta)x} = e^{\alpha x} e^{-i\beta x}$.

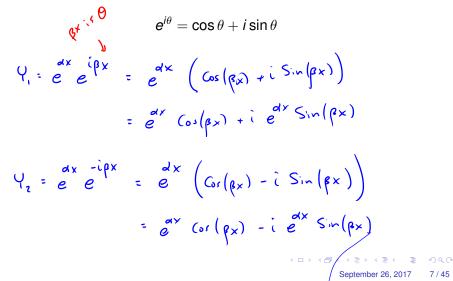
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Deriving the solutions Case III

Recall Euler's Formula:



Set

$$\begin{aligned}
& b_{1} = \frac{1}{2} \left((\gamma_{1} + \gamma_{2}) = \frac{1}{2} \left(2e^{4x} \cos(\beta \times) + \delta \right) \\
& = e^{4x} \cos(\beta \times)
\end{aligned}$$



Solve the ODE $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 6x = 0$ \sim $m^2 + 4m + 6 = 0$ Charc egn -4+ 5-8 $x_1 = e^{2t} C_{0s}(52t)$ -4 ± 252 i e, t Sin (J2t) -d sola is = -z ± 12 c x= c, e Cos(12+) + c, e Sin(12+) September 26, 2017 9/45

Higer Order Linear Constant Coefficient ODEs

- The same approach applies. For an nth order equation, we obtain an nth degree polynomial.
- Complex roots must appear in conjugate pairs (due to real coefficients) giving a pair of solutions e^{αx} cos(βx) and e^{αx} sin(βx).
- If a root m is repeated k times, we get k linearly independent solutions

$$e^{mx}$$
, xe^{mx} , x^2e^{mx} , ..., $x^{k-1}e^{mx}$

or in conjugate pairs cases 2k solutions

 $e^{\alpha x}\cos(\beta x), e^{\alpha x}\sin(\beta x), xe^{\alpha x}\cos(\beta x), xe^{\alpha x}\sin(\beta x), \dots,$

 $x^{k-1}e^{\alpha x}\cos(\beta x), x^{k-1}e^{\alpha x}\sin(\beta x)$

It may require a computer algebra system to find the roots for a high degree polynomial.

Example

Solve the ODE

y'''-4y'=0

 $m^{3} - 4m = 0$ Characheistic ean $m(m^2-4)=0$ m(m-2)(m+2) = 0M,=0, M2=2, M3=-2.

Several solution is $y = C_1 + C_2 e^{2x} + C_3 e^{-2x}$

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Example

 $m^{3} - 3m^{2} + 3m - 1 = 0$ Solve the ODE Charc. egh $(m-1)^{3} = 0$ y''' - 3y'' + 3y' - y = 0M=1, triple root $\mathcal{Y}_{1} = e^{\times}, \mathcal{Y}_{2} = \chi e^{\times}, \mathcal{Y}_{3} = \chi^{2} e^{\times}$ General solution $y=C, e^{x} + C_{2} \times e^{x} + C_{3} \times e^{x}$

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

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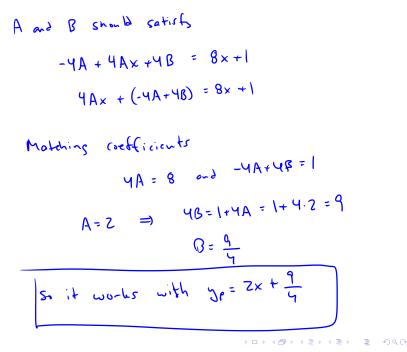
Motivating Example

Find a particular solution of the ODE

$$y'' - 4y' + 4y = 8x + 1$$

We'll guess that $\Im p$ matches the "form" of
 $g(x) = 8x + 1$. g is a line, so we'll guess
that $\Im p = Ax + B$ for some constants A and B .
 $\Im p' = A$, $\Im p'' = 0$
so $\Im p'' = 4\Im p' + 4\Im p = 0 - 4(A) + 4(Ax + B) = 8x + 1$

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Check $y_{p} = 2x + \frac{2}{4}, y_{p}' = 2, y_{p}'' = 0$ $y_{p}'' - 4y_{p}' + 4y_{p} = 0 - 4(2) + 4(2x + \frac{2}{4})$ = -8 + 8x + 9= 8x + 1

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