September 21 Math 3260 sec. 58 Fall 2017

Section 3.1: Introduction to Determinants

Recall that a 2×2 matrix is invertible if and only if the number called its **determinant** is nonzero. We had

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{21}a_{12}.$$

We wish to extend the concept of determinant to $n \times n$ matrices in general. And we wish to do so in such a way that invertibility holds if and only if the determinant is nonzero.

Determinant 3×3 case:

Suppose we start with a 3×3 invertible matrix. And suppose that $a_{11} \neq 0$. We can multiply the second and third rows by a_{11} and begin row reduction.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11}a_{21} & a_{11}a_{22} & a_{11}a_{23} \\ a_{11}a_{31} & a_{11}a_{32} & a_{11}a_{33} \end{bmatrix} \sim$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{13}a_{31} \end{bmatrix}$$

Determinant 3×3 case continued...

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{13}a_{31} \end{bmatrix}$$

If $A \sim I$, one of the entries in the $2,2$ or the $3,2$ position must be nonzero. Let's assume it is the $2,2$ entry. Continue row reduction to get

$$A \sim \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & 0 & a_{11}\Delta \end{bmatrix}.$$

Again, if A is invertible, it must be that the bottom right entry is nonzero. That is

$$\Delta \neq 0.$$

Note that if $\Delta = 0$, the rref of A is not I—A would be singular.

Determinant 3×3 case continued...

With a little rearrangement, we have

$$\begin{aligned}\Delta &= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + \\ &+ \quad a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\ \\ &= a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}\end{aligned}$$

The number Δ will be called the **determinant** of A.

Definitions: Minors

Let $n \geq 2$. For an $n \times n$ matrix A, let A_{ij} denote the $(n-1) \times (n-1)$ matrix obtained from A by deleting the i^{th} row and the j^{th} column of A.

For example, if

$$A = \begin{bmatrix} -1 & 3 & 2 & 0 \\ 4 & 4 & 0 & -3 \\ -2 & 1 & 7 & 2 \\ 3 & 0 & -1 & 6 \end{bmatrix} \quad \text{then} \quad A_{23} = \begin{bmatrix} -1 & 3 & 0 \\ -2 & 1 & 2 \\ 3 & 0 & 6 \end{bmatrix}.$$

Definition: The i, j^{th} **minor** of the $n \times n$ matrix A is the number

$$M_{ij} = \det(A_{ij}).$$

Definitions: Cofactor

Definition: Let A be an $n \times n$ matrix with $n \geq 2$. The i, j^{th} **cofactor** of A is the number

$$C_{ij} = (-1)^{i+j} M_{ij}.$$

Example: Find the three minors M_{11}, M_{12}, M_{13} and find the 3 cofactors C_{11}, C_{12}, C_{13} of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

$$A_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \qquad M_{11} = \det(A_{11})$$

$$= a_{22}a_{33} - a_{32}a_{23}$$

$$C_{11} = (-1)^{1+1} M_{11} = a_{22}a_{33} - a_{32}a_{23}$$

(Example Continued...)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

$$A_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$$

$$M_{12} = a_{21}a_{33} - a_{31}a_{23}$$

$$C_{12} = (-1)^{1+2} M_{12} = -(a_{21}a_{33} - a_{31}a_{23})$$

$$A_{13} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \qquad M_{13} = \det(A_{13}) = a_{21}a_{32} - a_{31}a_{22}$$

$$C_{13} = (-1)^{1+3} M_{13} = a_{21}a_{32} - a_{31}a_{22}$$

Observation:

Comparison with the determinant of the 3×3 matrix, we can note that

$$\Delta = a_{11}\det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

Definition: Determinant

For $n \geq 2$, the **determinant** of the $n \times n$ matrix $A = [a_{ij}]$ is the number

$$\begin{aligned}\det(A) &= a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} M_{1j}\end{aligned}$$

(Well call such a sum a **cofactor expansion.**)

Example: Evaluate the determinant

$$A = \begin{bmatrix} -1 & 3 & 0 \\ -2 & 1 & 2 \\ 3 & 0 & 6 \end{bmatrix}$$

$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

$= (-1)\,\det\left(\begin{bmatrix} 1 & 2 \\ 0 & 6 \end{bmatrix}\right) + 3(-1)\,\det\left(\begin{bmatrix} -2 & 2 \\ 3 & 6 \end{bmatrix}\right) + 0\cdot\det\left(\begin{bmatrix} -2 & 1 \\ 3 & 0 \end{bmatrix}\right)$

$= -1(1\cdot 6 - 0\cdot 2) - 3(-2\cdot 6 - 3\cdot 2) + 0$

$= -6 - 3(-18) = -6 + 54 = 48$

Theorem:

The determinant of an $n \times n$ matrix can be computed by cofactor expansion across any row or down any column.

Example: Find the determinant of the matrix

$$A = \begin{bmatrix} -1 & 3 & 4 & 0 \\ 0 & 0 & -3 & 0 \\ -2 & 1 & 2 & 2 \\ 3 & 0 & -1 & 6 \end{bmatrix}$$

we can do a cofactor expansion across row two.

$$\det(A) = \underbrace{a_{21}}_{=0} C_{21} + \underbrace{a_{22}}_{=0} C_{22} + a_{23} C_{23} + \underbrace{a_{24}}_{=0} C_{24}$$

$$A_{23} = \begin{bmatrix} -1 & 3 & 0 \\ -2 & 1 & 2 \\ 3 & 0 & 6 \end{bmatrix} \quad \text{so} \quad M_{23} = \det(A_{23}) = 48$$

So

$$\det(A) = a_{23}C_{23}$$

$$= (-3)(-1)^{2+3}M_{23}$$

$$= 3(48) = 144$$

Triangular Matrices

Definition:

The $n \times n$ matrix $A = [a_{ij}]$ is said to be **upper triangular** if $a_{ij} = 0$ for all $i > j$. It is said to be **lower triangular** if $a_{ij} = 0$ for all $j > i$. A matrix that is both upper and lower triangular is **diagonal**.

Theorem: For $n \geq 2$, the determinant of an $n \times n$ triangular matrix is the product of its diagonal entries. (i.e. if $A = [a_{ij}]$ is triangular, then $\det(A) = a_{11}a_{22}\cdots a_{nn}$.)

Example

$$A = \begin{bmatrix} -1 & 3 & 4 & 0 & 2 \\ 0 & 2 & -3 & 0 & -4 \\ 0 & 0 & 3 & 7 & 5 \\ 0 & 0 & 0 & -4 & 6 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

$\det(A) = (-1)\cdot 2 \cdot 3 \cdot (-4) \cdot 6$

$= 144$

$\det(A) = 7 \cdot 6 \cdot 2 \cdot 2 = 168$

$$A = \begin{bmatrix} 7 & 0 & 0 & 0 \\ 3 & 6 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 4 & 2 & 2 & 2 \end{bmatrix}$$

Section 3.2: Properties of Determinants

Theorem: Let A be an $n \times n$ matrix, and suppose the matrix B is obtained from A by performing a single elementary row operation[1]. Then

(i) If B is obtained by adding a multiple of a row of A to another row of A (row replacement), then

$$\det(B) = \det(A).$$

(ii) If B is obtained from A by swapping any pair of rows (row swap) , then

$$\det(B) = -\det(A).$$

(iii) If B is obtained from A by scaling any row by the constant k (scaling), then

$$\det(B) = k\det(A).$$

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[1] If "row" is replaced by "column" in any of the operations, the conclusions still follow.

Example: Compute the Determinant

$$\begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{vmatrix}$$

Let's call the matrix A and do row reduction to an ref B.

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 2 & 5 & -7 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{bmatrix}$$

$R_1 + R_4 \to R_4$

Changes

row swap factor -1

replacement no effect

replacement no effect

row swap factor -1

$$\begin{bmatrix} 2 & 5 & -7 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 1 \end{bmatrix}$$

$$-3R_2 + R_3 \to R_3$$

$$\begin{bmatrix} 2 & 5 & -7 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & -3 & 1 \end{bmatrix}$$

$$R_3 \leftrightarrow R_4$$

$$\begin{bmatrix} 2 & 5 & -7 & 3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix} = B$$

$$\det(B) = 2 \cdot 1 \cdot (-3) \cdot 5$$

$$= -30$$

$$\det(A) = \frac{\det(B)}{(-1)(-1)} = -30$$

Some Theorems:

Theorem: The $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$.

Theorem: For $n \times n$ matrix A, $\det(A^T) = \det(A)$.

Theorem: For $n \times n$ matrices A and B, $\det(AB) = \det(A)\det(B)$.

Example

Show that if A is an $n \times n$ invertible matrix, then

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Note $\det(I_n) = 1$. We have $I = AA^{-1}$, so

$$\det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1})$$

Hence $1 = \det(A)\det(A^{-1})$

So $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

Example

Let A be an $n \times n$ matrix, and suppose there exists invertible matrix P such that

$$B = P^{-1}AP.$$

Show that

$$\det(B) = \det(A).$$

$$\det(B) = \det(P^{-1}AP) = \det(P^{-1})\det(AP) = \underbrace{\det(P^{-1})\det(A)\det(P)}_{\text{real number factors}}$$

$$\Rightarrow \det(B) = \underbrace{\det(P^{-1})\det(P)}_{=1}\det(A).$$

$$\Rightarrow \det(B) = \det(A).$$

Section 4.1: Vector Spaces and Subspaces

Definition A **vector space** is a nonempty set V of objects called *vectors* together with two operations called *vector addition* and *scalar multiplication* that satisfy the following ten axioms: For all $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V, and for any scalars c and d

1. The sum $\mathbf{u}+\mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$ is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. There exists a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each vector $\mathbf{u}$ there exists a vector $-\mathbf{u}$ such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. For each scalar c, $c\mathbf{u}$ is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=d(c\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$

Remarks

- V is more accurately called a *real vector space* when we assume that the relevant scalars are the real numbers.
- Property 1. is that V is **closed** under (a.k.a. with respect to) vector addition.
- Property 6. is that V is **closed** under scalar multiplication.
- A vector space has the same basic *structure* as $\mathbb{R}^n$
- These are **axioms.** We assume (not "prove") that they hold for vector space V. However, they can be used to **prove or disprove** that a given set (with operations) is actually a vector space.

Examples of Vector Spaces

For an integer $n \geq 0$, $\mathbb{P}_n$ denotes the set of all polynomials with real coefficients of degree at most n. That is

$$\mathbb{P}_n = \{\mathbf{p}(t) = p_0 + p_1 t + \cdots + p_n t^n \mid p_0, p_1, \ldots, p_n \in \mathbb{R}\},$$

where addition[2] and scalar multiplication are defined by

$$(\mathbf{p}+\mathbf{q})(t) = \mathbf{p}(t) + \mathbf{q}(t) = (p_0+q_0) + (p_1+q_1)t + \cdots + (p_n+q_n)t^n,$$

$$(c\mathbf{p})(t) = c\mathbf{p}(t) = cp_0 + cp_1 t + \cdots + cp_n t^n.$$

[2] $\mathbf{q}(t) = q_0 + q_1 t + \cdots + q_n t^n$

Example

What is the zero vector $\mathbf{0}$ in $\mathbb{P}_n$?

$\vec{0}$ should satisfy $\vec{0}+\vec{p}=\vec{p}$ for each $\vec{p}$ in $\mathbb{P}_n$

Let $\vec{0}=a_0+a_1t+a_2t^2+\dots+a_nt^n$

$$(\vec{0}+\vec{p})(t)=\vec{0}(t)+\vec{p}(t)=(a_0+p_0)+(a_1+p_1)t+\dots+(a_n+p_n)t^n$$

$$=p_0+p_1t+p_2t^2+\dots+p_nt^n$$

This requires $a_i=0$ for each $i=0,\dots,n$.

$$\vec{0}=0+0t+0t^2+\dots+0t^n=0$$

Example

If $\mathbf{p}(t) = p_0 + p_1 t + \cdots + p_n t^n$, what is the vector $-\mathbf{p}$?

We know that $\vec{p} + (-\vec{p}) = \vec{0}$

Let $-\vec{p}(t) = b_0 + b_1 t + b_2 t^2 + \dots + b_n t^n$

$$(\vec{p} + (-\vec{p}))(t) = \vec{p}(t) + (-\vec{p})(t)$$
$$= (p_0 + b_0) + (p_1 + b_1) t + \dots + (p_n + b_n) t^n = \vec{0}$$

This requires $p_i + b_i = 0$ for each i, $b_i = -p_i$

$-\vec{p}(t) = -p_0 - p_1 t - \dots - p_n t^n$.

Examples of Vector Spaces

Let V be the set of all differentiable, real valued functions $f(x)$ defined for $-\infty < x < \infty$ with the property that

$$f(0) = 0.$$

Define vector addition and scalar multiplication in the standard way for functions—i.e.

$$(f+g)(x) = f(x) + g(x), \quad \text{and} \quad (cf)(x) = cf(x).$$

Example

Verify that properties 1. and 6. hold.

If f and g are differentiable then $(f+g)' = f' + g'$

and $(cf)' = cf'$. So $f+g$ and cf are

differentiable for any f, g in our set and

scalar c.

For f and g in our set, $f(0)=0$ and $g(0)=0$

so $(f+g)(0) = f(0) + g(0) = 0+0 = 0$.

The set is closed under vector addition.

Also
$$(cf)(0) = cf(0) = c \cdot 0 = 0.$$

So the set is closed under scalar multiplication.

A set that is not a Vector Space

Let $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, \mid x \le 0, y \le 0 \right\}$ with regular vector addition and scalar multiplication in $\mathbb{R}^2$. Note V is the third quadrant in the xy-plane.

(1) Does property 1. hold for V?

Let $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} u \\ v \end{bmatrix}$ in V. So $x \le 0$, $y \le 0$, $u \le 0$, and $v \le 0$.

$$\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x+u \\ y+v \end{bmatrix}$$

$x+u \le 0$ and

$y+v \le 0$

V is closed under vector addition.

A set that is not a Vector Space

Let $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, \mid x \leq 0, y \leq 0 \right\}$ with regular vector addition and scalar multiplication in $\mathbb{R}^2$. Note V is the third quadrant in the xy-plane.

(2) Does property 6. hold for V?

No, consider $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$ in V. Note

$(-1)\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ which is not in V.

Property 6 doesn't hold; V is not a vector space.