Sept. 28 Math 1190 sec. 51 Fall 2016

Section 3.2: Implicit Differentiation; Derivatives of the Inverse Trigonometric Functions

A relation—an equation involving two variables x and y, $F(x, y) = C$—such as

$$x^2 + y^2 = 16 \quad \text{or} \quad (x^2 + y^2)^3 = x^2$$

implies that y is defined to be one or more functions of x.

Given a relation, we can try to determine the derivative $\frac{dy}{dx}$.

Finding a Derivative Using Implicit Differentiation:

- Take the derivative of both sides of an equation with respect to the independent variable.
- Use all necessary rules for differenting powers, products, quotients, trig functions, exponentials, compositions, etc.
- Remember the chain rule for each term involving the dependent variable (e.g. mult. by $\frac{dy}{dx}$ as required).
- Use necessary algebra to isolate the desired derivative $\frac{dy}{dx}$.

Example

Find $\frac{dy}{dx}$.

$$y\,2^x + x\,2^y = 1$$

Recall

$$\frac{d}{dx} 2^x = 2^x \ln 2$$

$$\frac{d}{dx}\left(y\,2^x + x\,2^y\right) = \frac{d}{dx}(1)$$

products

$$\left(\frac{d}{dx} y\right) 2^x + y\left(\frac{d}{dx} 2^x\right) + \left(\frac{d}{dx} x\right) 2^y + x\left(\frac{d}{dx} 2^y\right) = 0$$

$$\frac{dy}{dx} 2^x + y\,2^x \ln 2 + 1 \cdot 2^y + x\,2^y \ln 2 \cdot \frac{dy}{dx} = 0$$

$$2^x \frac{dy}{dx} + x\,2^y \ln 2 \frac{dy}{dx} = -y\,2^x \ln 2 - 2^y$$

$$\left(2^x + x2^y \ln 2\right)\frac{dy}{dx} = -y2^x \ln 2 - 2^y$$

$$\boxed{\frac{dy}{dx} = \frac{-y2^x \ln 2 - 2^y}{2^x + x2^y \ln 2}}$$

Question

Find $\frac{dy}{dx}$.

$$e^y = xy$$

$$\frac{d}{dx} e^y = \frac{d}{dx}(xy)$$

$$e^y \cdot \frac{dy}{dx} = \underline{1} \cdot y + x \cdot \frac{dy}{dx}$$

$$e^y \frac{dy}{dx} - x\frac{dy}{dx} = y$$

$$(e^y - x)\frac{dy}{dx} = y \Rightarrow \frac{dy}{dx} = \frac{y}{e^y - x}$$

(a) $\dfrac{dy}{dx} = \dfrac{e^y - y}{x}$

(b) $\dfrac{dy}{dx} = \dfrac{e^y}{x}$

(c) $\dfrac{dy}{dx} = \dfrac{x}{e^y}$

d) $\dfrac{dy}{dx} = \dfrac{y}{e^y - x}$

Example

Find the equation of the line tangent to the graph of $x^3 + y^3 = 6xy$ at the point $(3, 3)$.

Note $(3,3)$ is on the curve since $3^3 + 3^3 = 54 = 6(3)(3)$.

We need a point and a slope. The point $(3,3)$ is given.

The slope $m_{tan} = \frac{dy}{dx}$ @ $(3,3)$.

Find $\frac{dy}{dx}$:

$$\frac{d}{dx}(x^3 + y^3) = \frac{d}{dx}(6xy)$$

$$3x^2 + 3y^2 \frac{dy}{dx} = 6\left(1 \cdot y + x\frac{dy}{dx}\right)$$

$$3x^2 + 3y^2 \frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$$

let's divide by 3

$$x^2 + y^2 \frac{dy}{dx} = 2y + 2x \frac{dy}{dx}$$

$$y^2 \frac{dy}{dx} - 2x \frac{dy}{dx} = 2y - x^2$$

$$(y^2 - 2x) \frac{dy}{dx} = 2y - x^2$$

$$\boxed{\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}}$$

The slope is the value of $\frac{dy}{dx}$ when $x = 3$ and $y = 3$.

$$m_{tan} = \frac{2(3) - (3^2)}{(3)^2 - 2(3)} = \frac{6-9}{9-6} = \frac{-3}{3} = -1$$

we have slope $m_{tan} = -1$ point $(x_1, y_1) = (3,3)$

$$y - 3 = -1(x - 3)$$

$$y = -x + 3 + 3 \implies \boxed{y = -x + 6}$$

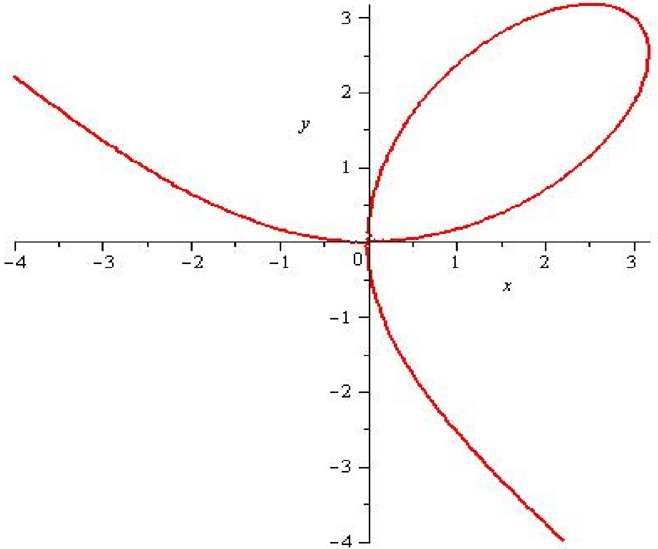

Figure: Folium of Descartes $x^3 + y^3 = 6xy$

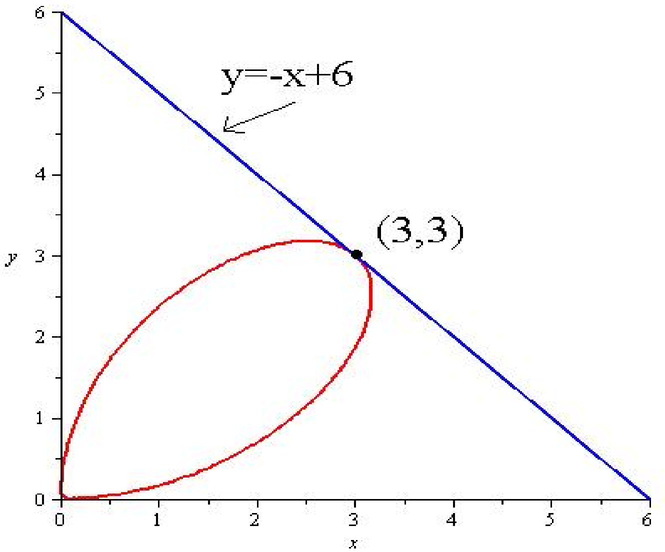

Figure: Folium of Descartes with tangent line at $(3, 3)$

Question

Find the equation of the line tangent to the graph of $y^2 + y + x^2 = 6$ at the point $(2, 1)$.

(a) $y = -6x + 13$

$\frac{dy}{dx} = \frac{-2x}{2y+1}$ $m = \frac{-2 \cdot 2}{2 \cdot 1 + 1} = \frac{-4}{3}$

(b) $y = -x + 3$

(c) $y = -\frac{4}{3}x + \frac{11}{3}$

$y - y_1 = m(x - x_1)$ for slope m point (x_1, y_1)

(d) $y = -\frac{5}{2}x + 6$

$y - 1 = \frac{-4}{3}(x - 2) \Rightarrow y = \frac{-4}{3}x + \frac{11}{3}$

The Power Rule: Rational Exponents

Let $y = x^{p/q}$ where p and q are integers. This can be written implicitly as

$$y^q = x^p.$$

Find $\frac{dy}{dx}$.

Using implicit differentiation

$$\frac{d}{dx} y^q = \frac{d}{dx} x^p$$

$$q y^{q-1} \frac{dy}{dx} = p x^{p-1}$$

$$\frac{dy}{dx} = \frac{p x^{p-1}}{q y^{q-1}}$$

$$\frac{dy}{dx} = \frac{p}{q}\,\frac{x^p \cdot x^{-1}}{y^q \cdot y^{-1}}$$

$$= \frac{p}{q}\,\frac{x^p}{y^q}\,x^{-1} \cdot y$$

Note

$x^p = y^q \Rightarrow \frac{x^p}{y^q} = 1$

and

$y = x^{p/q}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{p}{q} \cdot 1 \cdot x^{-1} \cdot x^{p/q}$$

$$\frac{dy}{dx} = \frac{p}{q}\,x^{\frac{p}{q}-1}$$

This turns out to be the same power rule!

The Power Rule: Rational Exponents

Theorem: If r is any rational number, then when x^r is defined, the function $y = x^r$ is differentiable and

$$\frac{d}{dx}x^r = rx^{r-1}$$

for all x such that x^{r-1} is defined.

For example $\frac{d}{dx}\sqrt{x} = \frac{d}{dx} x^{1/2} = \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2} x^{-1/2}$

$= \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}$

Examples

Evaluate

(a) $\frac{d}{dx}\sqrt[4]{x}$ $= \frac{d}{dx} x^{1/4} = \frac{1}{4} x^{\frac{1}{4}-1} = \frac{1}{4} x^{-3/4} = \frac{1}{4x^{3/4}} = \frac{1}{4\sqrt[4]{x^3}}$

(b) $\frac{d}{dv}\csc(\sqrt{v})$ $= -\csc(\sqrt{v})\cot(\sqrt{v}) \cdot \frac{1}{2\sqrt{v}} = \frac{-\csc(\sqrt{v})\cot(\sqrt{v})}{2\sqrt{v}}$

outside

root is inside

derivative of outside

derivative of inside

Question

Find $f'(x)$ where $f(x) = \sqrt[3]{2x}$.

$f(x) = (2x)^{\frac{1}{3}}$

(a) $f'(x) = \frac{1}{3}(2x)^{-2/3}$

$f'(x) = \frac{1}{3}(2x)^{-2/3} \cdot (2)$

$= \frac{2}{3}(2x)^{-2/3}$

(b) $f'(x) = \frac{2}{3}x^{-1/3}$

$= \frac{2}{3(2x)^{2/3}} = \frac{2}{3\sqrt[3]{(2x)^2}}$

(c) $f'(x) = \frac{2}{3\sqrt[3]{4x^2}}$

$= \frac{2}{3\sqrt[3]{4x^2}}$

(d) $f'(x) = \frac{2}{3}(x)^{-2/3}$