Sept. 28 Math 1190 sec. 52 Fall 2016

Section 3.2: Implicit Differentiation; Derivatives of the Inverse Trigonometric Functions

A relation—an equation involving two variables x and y, $F(x,y) = C$—such as

$$x^2 + y^2 = 16 \quad \text{or} \quad (x^2 + y^2)^3 = x^2$$

implies that y is defined to be one or more functions of x.

Given a relation, we can try to determine the derivative $\frac{dy}{dx}$.

Finding a Derivative Using Implicit Differentiation:

- Take the derivative of both sides of an equation with respect to the independent variable.
- Use all necessary rules for differenting powers, products, quotients, trig functions, exponentials, compositions, etc.
- Remember the chain rule for each term involving the dependent variable (e.g. mult. by $\frac{dy}{dx}$ as required).
- Use necessary algebra to isolate the desired derivative $\frac{dy}{dx}$.

Example

Find $\frac{dy}{dx}$.

$$y\,2^x + x\,2^y = 1$$

Recall

$$\frac{d}{dx} 2^x = 2^x \ln 2$$

$$\frac{d}{dx} fg = f'g + fg'$$

$$\frac{d}{dx}\left(y\,2^x + x\,2^y\right) = \frac{d}{dx}(1)$$

products

$$\frac{d}{dx}\left(y\,2^x\right) + \frac{d}{dx}\left(x\,2^y\right) = 0$$

$$\left(\frac{d}{dx} y\right) 2^x + y\left(\frac{d}{dx} 2^x\right) + \left(\frac{d}{dx} x\right) 2^y + x\left(\frac{d}{dx} 2^y\right) = 0$$

$$\frac{dy}{dx} 2^x + y\,2^x \ln 2 + 1 \cdot 2^y + x\,2^y \ln 2 \cdot \frac{dy}{dx} = 0$$

$$2^x \frac{dy}{dx} + x\,2^y \ln 2 \,\frac{dy}{dx} = -y\,2^x \ln 2 - 2^y$$

$$\left(2^x + x\,2^y \ln 2\right)\frac{dy}{dx} = -y\,2^x \ln 2 - 2^y$$

$$\frac{dy}{dx} = \frac{-y\,2^x \ln 2 - 2^y}{2^x + x\,2^y \ln 2}$$

Question

$$\frac{d}{dx}xy = \left(\frac{d}{dx}x\right)y + x\left(\frac{d}{dx}y\right)$$

Find $\frac{dy}{dx}$.

$$e^y = xy$$

$$\frac{d}{dx}e^{f(x)} = e^{f(x)} \cdot f'(x)$$

$$\frac{d}{dx}e^y = \frac{d}{dx}(xy)$$

product rule

$$e^y \frac{dy}{dx} = 1 \cdot y + x \cdot \frac{dy}{dx}$$

$$e^y \frac{dy}{dx} - x\frac{dy}{dx} = y$$

$$(e^y - x)\frac{dy}{dx} = y \Rightarrow \frac{dy}{dx} = \frac{y}{e^y - x}$$

(a) $\frac{dy}{dx} = \frac{e^y - y}{x}$

(b) $\frac{dy}{dx} = \frac{e^y}{x}$

(c) $\frac{dy}{dx} = \frac{x}{e^y}$

d) $\frac{dy}{dx} = \frac{y}{e^y - x}$

Example

Find the equation of the line tangent to the graph of $x^3 + y^3 = 6xy$ at the point $(3, 3)$.

Note that $(3,3)$ is on the graph since $3^3 + 3^3 = 54 = 6(3)(3)$

We need a point and a slope. The point $(3,3)$ is given.

The slope $m_{tan} = \frac{dy}{dx}$ @ $(3,3)$.

Find $\frac{dy}{dx}$: $\frac{d}{dx}(x^3 + y^3) = \frac{d}{dx}(6xy)$

$$3x^2 + 3y^2 \cdot \frac{dy}{dx} = 6\left(1 \cdot y + x \cdot \frac{dy}{dx}\right)$$

$$3x^2 + 3y^2 \frac{dy}{dx} = 6y + 6x \frac{dy}{dx}$$

Divide by 3

$$x^2 + y^2 \frac{dy}{dx} = 2y + 2x \frac{dy}{dx}$$

$$y^2 \frac{dy}{dx} - 2x \frac{dy}{dx} = 2y - x^2$$

$$(y^2 - 2x) \frac{dy}{dx} = 2y - x^2$$

$$\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}$$

$$m_{tan} = \frac{dy}{dx} @ (3,3) \quad , \quad m_{tan} = \frac{2 \cdot 3 - 3^2}{3^2 - 2 \cdot 3}$$

$$m_{tan} = \frac{6-9}{9-6} = \frac{-3}{3} = -1$$

our point $(x_0, y_0) = (3,3)$ and slope $m_{tan} = -1$

$$y - 3 = -1\,(x-3)$$

$$y = -x + 3 + 3$$

$$\Rightarrow \boxed{y = -x + 6}$$

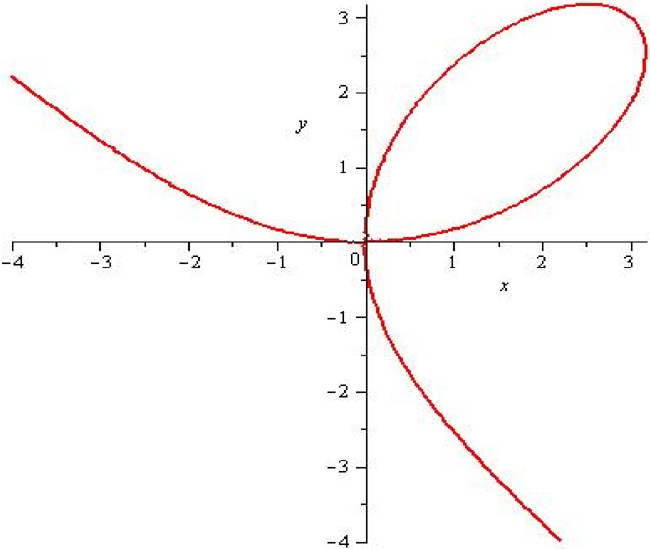

Figure: Folium of Descartes $x^3 + y^3 = 6xy$

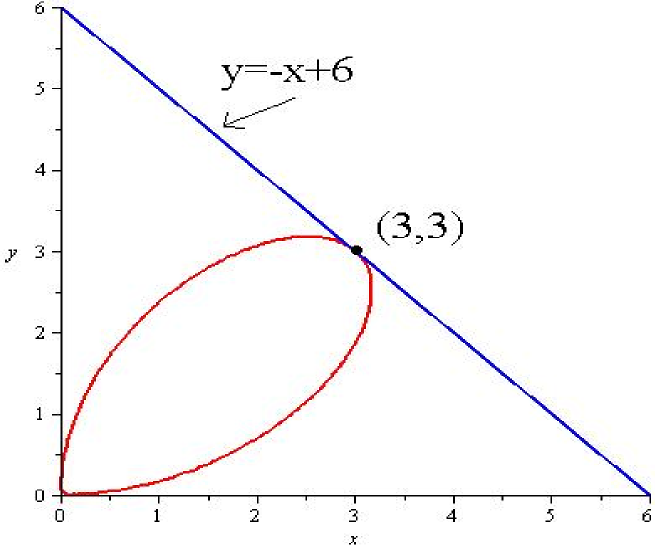

Figure: Folium of Descartes with tangent line at $(3, 3)$

Question

Find the equation of the line tangent to the graph of $y^2 + y + x^2 = 6$ at the point $(2, 1)$.

$$\frac{dy}{dx} = \frac{-2x}{2y+1}$$

(a) $y = -6x + 13$

(b) $y = -x + 3$

$$m = \frac{-2 \cdot 2}{2 \cdot 1 + 1} = \frac{-4}{3}$$

(c) $y = -\frac{4}{3}x + \frac{11}{3}$

(d) $y = -\frac{5}{2}x + 6$

The Power Rule: Rational Exponents

Let $y = x^{p/q}$ where p and q are integers. This can be written implicitly as

$$y^q = x^p.$$

Find $\frac{dy}{dx}$.

Use implicit differentiation

$$\frac{d}{dx} y^q = \frac{d}{dx} x^p$$

$$q y^{q-1} \frac{dy}{dx} = p x^{p-1}$$

$$\frac{dy}{dx} = \frac{p x^{p-1}}{q y^{q-1}}$$

$$\frac{dy}{dx} = \frac{p\,x^{p} \cdot x^{-1}}{q\,y^{q} \cdot y^{-1}}$$

$$= \frac{p}{q}\,\frac{x^{p}}{y^{q}}\,x^{-1}\,y$$

$$= \frac{p}{q}\,(1)\,x^{-1}\,x^{\frac{p}{q}}$$

$$\frac{dy}{dx} = \frac{p}{q}\,x^{\frac{p}{q}-1}$$

Recall

$y = x^{p/q}$ and

$y^{q} = x^{p}$ so

$1 = \frac{x^{p}}{y^{q}}$

This is our same power rule.

The Power Rule: Rational Exponents

Theorem: If r is any rational number, then when x^r is defined, the function $y = x^r$ is differentiable and

$$\frac{d}{dx}x^r = rx^{r-1}$$

for all x such that x^{r-1} is defined.

For example

$$\frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{1/2} = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}$$

Examples

Evaluate

(a) $\frac{d}{dx}\sqrt[4]{x}$ $= \frac{d}{dx} x^{1/4} = \frac{1}{4} x^{1/4-1} = \frac{1}{4} x^{-3/4} = \frac{1}{4x^{3/4}} = \frac{1}{4\sqrt[4]{x^3}}$

(b) $\frac{d}{dv}\csc(\sqrt{v})$ $= -\csc(\sqrt{v})\cot(\sqrt{v}) \cdot \frac{1}{2\sqrt{v}} = \frac{-\csc(\sqrt{v})\cot(\sqrt{v})}{2\sqrt{v}}$

outside cosecant

inside square root

Question

Find $f'(x)$ where $f(x) = \sqrt[3]{2x}$.

$f(x) = (2x)^{\frac{1}{3}}$

(a) $f'(x) = \frac{1}{3}(2x)^{-2/3}$

$f'(x) = \frac{1}{3}(2x)^{\frac{1}{3}-1} \cdot (2)$

$= \frac{1}{3}(2x)^{-\frac{2}{3}} \cdot 2$

(b) $f'(x) = \frac{2}{3}x^{-1/3}$

$= \frac{2}{3(2x)^{2/3}}$

(c) $f'(x) = \frac{2}{3\sqrt[3]{4x^2}}$

$= \frac{2}{3\sqrt[3]{(2x)^2}} = \frac{2}{3\sqrt[3]{4x^2}}$

(d) $f'(x) = \frac{2}{3}(x)^{-2/3}$