September 28 Math 2306 sec 51 Fall 2015

Section 4.4: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

Motivating Example

Find a particular solution of the ODE

$$y'' - 4y' + 4y = 8x + 1$$

We'll make a guess as to the form of y_p, and see if we can make it work.

Since $g(x) = 8x+1$ is a line, perhaps y_p is a line.

Suppose $y_p = Ax + B$

Plug y_p into the DE

$$y_p = Ax + B, \quad y_p' = A, \quad y_p'' = 0$$

we need $\quad y_p'' - 4y_p' + 4y_p = 8x + 1$

$$0 - 4(A) + 4(Ax + B) = 8x + 1$$

$$4Ax + 4B - 4A = 8x + 1$$

Match coefficients

$$4A = 8 \quad \Rightarrow \quad A = 2$$

$$-4A + 4B = 1 \quad \Rightarrow \quad 4B = 1 + 4A = 1 + 4 \cdot 2 = 9$$

$$B = \frac{9}{4}$$

we found coefficents that work so

$$y_p = 2x + \frac{9}{4}.$$

The Method: Assume y_p has the same **form** as $g(x)$

$$y'' - y' = 4e^{-5x}$$

Here, $g(x) = 4e^{-5x}$, a constant times e^{-5x}.

So let's see if

$$y_p = Ae^{-5x}$$

$$y_p' = -5Ae^{-5x}$$

$$y_p'' = 25Ae^{-5x}$$

If y_p is a solution then

$$y_p'' - y_p' = 4e^{-5x}$$

$$25Ae^{-5x} - (-5Ae^{-5x}) = 4e^{-5x}$$

$$25Ae^{-5x} + 5Ae^{-5x} = 4e^{-5x}$$

$$30Ae^{-5x} = 4e^{-5x}$$

$$30A = 4$$

$$A = \frac{2}{15}$$

Matching coefficients

$$y_p = \frac{2}{15}e^{-5x}$$

Make the form general

$$y'' - 4y' + 4y = 16x^2$$

Here $g(x) = 16x^2$ a constant times x^2.

Perhaps $y_p = Ax^2$.

$y_p = Ax^2$, $y_p' = 2Ax$, $y_p'' = 2A$

$y_p'' - 4y_p' + 4y_p = 16x^2$

$2A - 4(2Ax) + 4Ax^2 = 16x^2$

$$4Ax^2 - 8Ax + 2A = 16x^2 + 0x + 0$$

Match coefficients

$$\left.\begin{aligned} 4A &= 16 \\ -8A &= 0 \\ 2A &= 0 \end{aligned}\right\} \Rightarrow$$ requires $A=4$ and $A=0$ which isn't possible.

Let's try again. $g(x) = 16x^2$ is a quadratic.

Suppose $y_p = Ax^2 + Bx + C$

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

$$y_p'' - 4y_p' + 4y_p = 16x^2$$

$$2A - 4(2Ax+B) + 4(Ax^2+Bx+C) = 16x^2$$

$$4Ax^2 + (-8A+4B)x + (2A-4B+4C) = 16x^2 + 0x + 0$$

$$4A = 16 \Rightarrow A = 4$$

$$-8A+4B = 0 \Longrightarrow 4B = 8A \Rightarrow B = 2A = 2\cdot 4 = 8$$

$$2A - 4B + 4C = 0 \Rightarrow 4C = 4B - 2A = 4\cdot 8 - 2\cdot 4 = 24$$

$$4C = 24 \quad C = 6$$

This worked giving

$$\boxed{y_p = 4x^2 + 8x + 6}$$

General Form: sines and cosines

$$y'' - y' = 20\sin(2x)$$

$g(x) = 20\sin(2x)$ constant times $\sin(2x)$

Perhaps $\quad y_p = A\sin(2x)$

$$y_p' = 2A\cos(2x)$$

$$y_p'' = -4A\sin(2x)$$

$$y_p'' - y_p' = 20\sin(2x)$$

$$-4A\,\sin(2x) - 2A\,\cos(2x) = 20\,\sin(2x)$$

$$\underline{\underline{-4A}}\,\sin(2x) - \underline{\underline{2A}}\,\cos(2x) = \underline{\underline{20}}\,\sin(2x) + \underline{\underline{0}}\,\cos(2x)$$

$$\left.\begin{aligned} -4A &= 20 \\ -2A &= 0 \end{aligned}\right\} \Rightarrow$$

$A = -5$ and $A = 0$ which can't be

Try again with

$$y_p = A\sin(2x) + B\cos(2x)$$

$$y_p' = 2A\cos(2x) - 2B\sin(2x)$$

$$y_p'' = -4A\sin(2x) - 4B\cos(2x)$$

$$y_p'' - y_p' = 20\sin(2x)$$

$$-4A\sin(2x) - 4B\cos(2x) - \left(2A\cos(2x) - 2B\sin(2x)\right) = 20\sin(2x)$$

$$\underline{\underline{(-4A+2B)}}\sin(2x) + (\underline{\underline{-2A-4B}})\cos(2x) = \underline{\underline{20}}\sin(2x) + \underline{\underline{0}}\cdot\cos(2x)$$

$$-4A + 2B = 20$$

$$-2A - 4B = 0 \implies 4B = -2A, \quad A = -2B$$

$$-4A + 2B = 20$$

$$-4(-2B) + 2B = 20 \implies 10B = 20 \quad B = 2$$

$A = -2B = -4$

This worked giving

$$\boxed{y_p = -4\sin(2x) + 2\cos(2x).}$$

Examples of Forms of y_p based on g (Trial Guesses)

(a) $g(x) = 1$ (or really any constant)

g is a constant function

$y_p = A$

(b) $g(x) = x - 7$

g is 1st degree polynomial

$y_p = Ax + B$

(c) $g(x) = 5x$

g is a 1st degree polynomial

$y_p = Ax + B$

(d) $g(x) = 3x^3 - 5$

g is a 3rd degree polynomial

$y_p = Ax^3 + Bx^2 + Cx + D$

More Trial Guesses

(e) $g(x) = xe^{3x}$ 1st degree polynomial times e^{3x}

$y_p = (Ax + B)e^{3x}$

(f) $g(x) = \cos(7x)$ Constants times $\sin(7x)$ and $\cos(7x)$

$y_p = A\cos(7x) + B\sin(7x)$

(g) $g(x) = \sin(2x) - \cos(4x)$ constants times sines and cosines of $2x$ and $4x$

$y_p = A\sin(2x) + B\cos(2x) + C\sin(4x) + D\cos(4x)$

(h) $g(x) = x^2\sin(3x)$
a quadratic times sine and cosine $3x$

$y_p = (Ax^2 + Bx + C)\sin(3x) + (Dx^2 + Ex + F)\cos(3x)$