

Section 4.4: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- ▶ polynomials,
- ▶ exponentials,
- ▶ sines and/or cosines,
- ▶ and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

Motivating Example

Find a particular solution of the ODE

$$y'' - 4y' + 4y = 8x + 1$$

$f(x) = 8x + 1$ is a line, a 1st degree polynomial
Let's guess that y_p has the same form.

Perhaps $y_p = Ax + B$

This is supposed to solve the DE.

$$y_p' = A, \quad y_p'' = 0$$

$$y_p'' - 4y_p' + 4y_p = 8x + 1$$

$$0 - 4(A) + 4(Ax+B) = 8x + 1$$

$$\underline{4Ax} + \underline{(-4A+4B)} = \underline{8x} + \underline{1}$$

Match coefficients

$$4A = 8 \Rightarrow A = 2$$

$$-4A + 4B = 1 \Rightarrow 4B = 1 + 4A = 1 + 8 = 9$$

$$B = \frac{9}{4}$$

so

$$y_p = 2x + \frac{9}{4}$$

The Method: Assume y_p has the same form as $g(x)$

$$y'' - y' = 4e^{-5x}$$

$g(x) = 4e^{-5x}$ which is a constant times e^{-5x}

Perhaps $y_p = A e^{-5x}$

$$y_p' = -5A e^{-5x}$$

$$y_p'' = 25A e^{-5x}$$

$$y_p'' - y_p' = 4e^{-sx}$$

$$25Ae^{-sx} - (-SAe^{-sx}) = 4e^{-sx}$$

$$\underline{30A} e^{-sx} = \underline{4} e^{-sx}$$

Matching coefficients

$$30A = 4 \Rightarrow A = \frac{2}{15}$$

So $y_p = \frac{2}{15} e^{-sx}$.

Make the form general

$$y'' - 4y' + 4y = 16x^2$$

$g(x) = 16x^2$ this is a constant times x^2

perhaps $y_p = Ax^2$

$$y_p' = 2Ax$$

$$y_p'' = 2A$$

$$y_p'' - 4y_p' + 4y_p = 16x^2$$

$$2A - 4(2Ax) + 4(Ax^2) = 16x^2$$

$$\begin{matrix} 4Ax^2 \\ \underline{-} \\ \underline{=}\end{matrix} - \begin{matrix} 8Ax \\ \underline{=} \end{matrix} + \begin{matrix} 2A \\ \underline{=} \end{matrix} = \begin{matrix} 16x^2 \\ \underline{=} \end{matrix} + \begin{matrix} 0x \\ \underline{=} \end{matrix} + \begin{matrix} 0 \\ \underline{=} \end{matrix}$$

we need to match coefficients

$$\left. \begin{array}{l} 4A = 16 \\ -8A = 0 \\ 2A = 0 \end{array} \right\} \begin{array}{l} \text{This requires } A=4 \text{ and } A=0 \\ \text{which can't be.} \end{array}$$

Try again noting that $g(x)$ is a 2nd degree polynomial.

Perhaps $y_p = Ax^2 + Bx + C$

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

$$y_p'' - 4y_p' + 4y_p = 16x^2$$

$$2A - 4(2Ax + B) + 4(Ax^2 + Bx + C) = 16x^2$$

$$\underline{\underline{4Ax^2}} + \underline{\underline{(-8A + 4B)x}} + \underline{\underline{(2A - 4B + 4C)}} = \underline{\underline{16x^2}} + \underline{0x} + \underline{0}$$

Match Coefficients

$$4A = 16 \Rightarrow A = 4$$

$$-8A + 4B = 0 \Rightarrow 4B = 8A \Rightarrow B - 2A = 2 \cdot 4 = 8$$

$$2A - 4B + 4C = 0 \Rightarrow 4C = 4B - 2A$$

$$= 4 \cdot 8 - 2 \cdot 4 = 24$$

$$C = 6$$

$$\text{So } y_p = 4x^2 + 8x + 6.$$

General Form: sines and cosines

$$y'' - y' = 20 \sin(2x)$$

$f(x) = 20 \sin(2x)$ is a constant times $\sin(2x)$

Perhaps $y_p = A \sin(2x)$

$$y_p' = 2A \cos(2x)$$

$$y_p'' = -4A \sin(2x)$$

$$y_p'' - y_p' = 20 \sin(2x)$$

$$\underline{-4A \sin(2x)} - \underline{2A} \cos(2x) = \underline{20} \sin(2x) + \underline{0} \cdot \cos(2x)$$

Matching coefficients

$$\begin{cases} -4A = 20 \\ -2A = 0 \end{cases} \quad \begin{array}{l} \text{not solvable because it} \\ \text{requires } A = -5 \text{ and } A = 0. \end{array}$$

Try again thinking of y as a
linear combo of $\sin(2x)$ and $\cos(2x)$

$$\text{Try } y_p = A \sin(2x) + B \cos(2x)$$

$$y_p' = 2A \cos(2x) - 2B \sin(2x)$$

$$y_p'' = -4A \sin(2x) - 4B \cos(2x)$$

$$y_p'' - y_p' = 20 \sin(2x)$$

$$-4A \sin(2x) - 4B \cos(2x) - 2A \cos(2x) + 2B \sin(2x) = 20 \sin(2x)$$

$$\underline{(-4A + 2B) \sin(2x)} + \underline{(-2A - 4B) \cos(2x)} = \underline{20} \sin(2x) + 0 \cdot \cos(2x)$$

$$-4A + 2B = 20$$

$$-2A - 4B = 0$$

$$-4A + 2B = 20$$

$$4A + 8B = 0$$

add

$$\underline{10B = 20} \Rightarrow B = 2$$

$$4A = -8B \Rightarrow A = -2B = -4$$

$$y_p = -4 \sin(2x) + 2 \cos(2x)$$

Examples of Forms of y_p based on g (Trial Guesses)

(a) $g(x) = 1$ (or really any constant) Constant

$$y_p = A$$

(b) $g(x) = x - 7$ 1st degree polynomial

$$y_p = Ax + B$$

(c) $g(x) = 5x$ Same as (b) $y_p = Ax + B$

(d) $g(x) = 3x^3 - 5$ 3rd degree polynomial

$$y_p = Ax^3 + Bx^2 + Cx + D$$

More Trial Guesses

(e) $g(x) = xe^{3x}$ 1st degree polynomial times e^{3x}

$$y_p = (Ax + \beta) e^{3x}$$

(f) $g(x) = \cos(7x)$ linear combo of $\sin(7x)$ and $\cos(7x)$

$$y_p = A \sin(7x) + B \cos(7x)$$

(g) $g(x) = \sin(2x) - \cos(4x)$ linear combos of sine/cosines of $2x$ and $4x$

$$y_p = A \sin(2x) + B \cos(2x) + C \sin(4x) + D \cos(4x)$$

(h) $g(x) = x^2 \sin(3x)$ Quadratic and linear combo of sines/cosines

$$y_p = (Ax^2 + Bx + C) \sin(3x) + (Dx^2 + Ex + F) \cos(3x)$$