September 28 Math 2306 sec. 56 Fall 2017

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

Motivating Example

Find a particular solution of the ODE

$$y'' - 4y' + 4y = 8x + 1$$

Note $g(x) = 8x+1$ is a 1^{st} degree polynomial (a.k.a a line).

Let's guess that y_p is also a 1^{st} degree polynomial.

$y_p = Ax + B$ for some numbers A and B.

Let's substitute $y_p' = A$, $y_p'' = 0$

$$y_p'' - 4y_p' + 4y_p = 0 - 4(A) + 4(Ax+B) = 8x+1$$

This requires

$$-4A + 4Ax + 4B = 8x + 1$$

$$4Ax + (-4A + 4B) = 8x + 1$$

Matching coefficients, we get

$$4A = 8 \quad \text{and} \quad -4A + 4B = 1$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$A = 2 \qquad\qquad 4B = 1 + 4A = 1 + 4 \cdot 2 = 9$$

$$B = \frac{9}{4}$$

So $y_p = 2x + \frac{9}{4}$

Let's verify that $y_p = 2x + \frac{9}{4}$ does solve

$$y'' - 4y' + 4y = 8x + 1$$

$y_p' = 2$, $y_p'' = 0$

on substitution

$$0 - 4(2) + 4\left(2x + \frac{9}{4}\right) =$$

$$-8 + 8x + 9 = 8x + 1$$

The Method: Assume y_p has the same **form** as $g(x)$

$$y'' - 4y' + 4y = 6e^{-3x}$$

Let's guess that $y_p = Ae^{-3x}$. We'll sub into the DE

$y_p' = -3Ae^{-3x}$, $y_p'' = 9Ae^{-3x}$

$$y_p'' - 4y_p' + 4y_p = 9Ae^{-3x} - 4(-3Ae^{-3x}) + 4(Ae^{-3x}) = 6e^{-3x}$$

$$9Ae^{-3x} + 12Ae^{-3x} + 4Ae^{-3x} = 6e^{-3x}$$

$$25Ae^{-3x} = 6e^{-3x}$$

This holds if $A = \frac{6}{25}$. $\boxed{\text{So } y_p = \frac{6}{25}e^{-3x}}$

Make the form general

$$y'' - 4y' + 4y = 16x^2$$

What if we assume $y_p = Ax^2$. Then

$y_p' = 2Ax$, $y_p'' = 2A$. Substitute

$$y_p'' - 4y_p' + 4y_p = 2A - 4(2Ax) + 4Ax^2 = 16x^2$$

This gives

$$4Ax^2 - 8Ax + 2A = 16x^2$$

Matching coefficients

$$4A = 16, \quad -8A = 0, \quad \text{and} \quad 2A = 0$$

$A = 4$ and $A = 0$! impossible!

We didn't account for an x or constant term.

If we consider $g(x) = 16x^2 = 16x^2 + 0x + 0$, we make the better assumption $y_p = Ax^2 + Bx + C$.

$$y_p' = 2Ax + B, \quad y_p'' = 2A$$

$$y_p'' - 4y_p' + 4y_p = 2A - 4(2Ax+B) + 4(Ax^2+Bx+C) = 16x^2$$

$$4Ax^2 + (-8A+4B)x + (2A-4B+4C) = 16x^2 + 0x + 0$$

Matching coefficients

$$4A = 16, \quad -8A + 4B = 0, \quad 2A - 4B + 4C = 0$$

$$A = 4, \quad 4B = 8A, \; B = 2A = 8 \quad 4C = -2A + 4B$$

$$C = \frac{-1}{2}A + B = -2 + 8 = 6$$

$$\boxed{\text{so} \quad y_p = 4x^2 + 8x + 6}$$

General Form: sines and cosines

$$y'' - y' = 20\sin(2x)$$

If we assume that $y_p = A\sin(2x)$, taking two derivatives would lead to the equation

$$-4A\sin(2x) - 2A\cos(2x) = 20\sin(2x).$$

This would require (matching coefficients of sines and cosines)

$$-4A = 20 \quad \text{and} \quad -2A = 0.$$

This is impossible as it would require $-5 = 0$!

General Form: sines and cosines

We must think of our equation $y'' - y' = 20\sin(2x)$ as

$$y'' - y' = 20\sin(2x) + 0\cos(2x).$$

The correct format for y_p is

$$y_p = A\sin(2x) + B\cos(2x).$$

when guessing at y_p, assume y_p contains all terms in $g(x)$ and all possible derivatives of those terms.

Examples of Forms of y_p based on g (Trial Guesses)

(a) $g(x) = 1$ (or really any constant)

$y_p = A$ Constant

(b) $g(x) = x - 7$

$y_p = Ax + B$ 1st degree Poly.

Examples of Forms of y_p based on g (Trial Guesses)

(c) $g(x) = 5x^2$

$y_p = Ax^2 + Bx + C$ 2nd degree poly.

(d) $g(x) = 3x^3 - 5$

$y_p = Ax^3 + Bx^2 + Cx + D$ 3rd degree poly.

Examples of Forms of y_p based on g (Trial Guesses)

(e) $g(x) = xe^{3x}$

$$y_p = (Ax+B)e^{3x}$$

1st degree poly. times e^{3x}

(f) $g(x) = \cos(7x)$

$$y_p = A\cos(7x) + B\sin(7x)$$

Examples of Forms of y_p based on g (Trial Guesses)

(g) $g(x) = \sin(2x) - \cos(4x)$

$$y_p = A\sin(2x) + B\cos(2x) + C\sin(4x) + D\cos(4x)$$

(h) $g(x) = x^2\sin(3x)$

$$y_p = (Ax^2 + Bx + C)\sin(3x) + (Dx^2 + Ex + F)\cos(3x).$$

2^{nd} degree poly times $\sin(3x)$ and $\cos(3x)$