# September 28 Math 2306 sec. 57 Fall 2017

## Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where $g$ comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

# Motivating Example

Find a particular solution of the ODE

$$y'' - 4y' + 4y = 8x + 1$$

We made a *guess* that $y_p = Ax + B$, a first order polynomial, because that's what $g(x)$ is. Substituting this guess into the ODE, we found that it works if $A = 2$ and $B = \frac{9}{4}$. So we did find a particular solution

$$y_p = 2x + \frac{9}{4}.$$

# The Method: Assume $y_p$ has the same **form** as $g(x)$

$$y'' - 4y' + 4y = 6e^{-3x}$$

$g(x) = 6e^{-3x}$ is a constant times $e^{-3x}$. Let's assume that

$y_p = Ae^{-3x}$. We substitute into the ODE.

$y_p' = -3Ae^{-3x}, \quad y_p'' = 9Ae^{-3x}$

$$y_p'' - 4y_p' + 4y_p = 9Ae^{-3x} - 4(-3Ae^{-3x}) + 4(Ae^{-3x}) = 6e^{-3x}$$

$$\Rightarrow \quad 9Ae^{-3x} + 12Ae^{-3x} + 4Ae^{-3x} = 6e^{-3x}$$

$$25Ae^{-3x} = 6e^{-3x} \quad \Rightarrow \quad A = \frac{6}{25}$$

$$\boxed{\text{Hence } y_p = \frac{6}{25}e^{-3x}}$$

# Make the form general

$$y'' - 4y' + 4y = 16x^2$$

$g(x) = 16x^2$ which is a constant times $x^2$. Perhaps

$y_p = Ax^2$. We can substitute $y_p' = 2Ax$, $y_p'' = 2A$

$$y_p'' - 4y_p' + 4y_p = 16x^2$$

$$2A - 4(2Ax) + 4(Ax^2) = 16x^2$$

$$4Ax^2 - 8Ax + 2A = 16x^2$$

Match coefficients

$$4A = 16, \quad -8A = 0, \quad 2A = 0$$

$\Rightarrow$ $A = 4$ and $A = 0$ not possible!

Let's consider $g(x) = 16x^2$ to be a 2nd degree polynomial

and suppose $y_p = Ax^2 + Bx + C$.

$y_p' = 2Ax + B$, $y_p'' = 2A$

$$y_p'' - 4y_p' + 4y_p = 16x^2$$

$$2A - 4(2Ax + B) + 4(Ax^2 + Bx + C) = 16x^2$$

$$4Ax^2 + (-8A + 4B)x + (2A - 4B + 4C) = 16x^2 + 0x + 0$$

Match Coefficients

$4A = 16$, $\quad -8A + 4B = 0 \quad 2A - 4B + 4C = 0$

$\Downarrow \qquad \Downarrow \qquad \Downarrow$

$A = 4$

$4B = 8A$

$B = 2A = 2\cdot 4 = 8$

$4C = -2A + 4B$

$C = \frac{-1}{2}A + B$

$= -2 + 8 = 6$

$$\boxed{\text{So} \quad y_p = 4x^2 + 8x + 6}$$

# General Form: sines and cosines

$$y'' - y' = 20\sin(2x)$$

If we assume that $y_p = A\sin(2x)$, taking two derivatives would lead to the equation

$$-4A\sin(2x) - 2A\cos(2x) = 20\sin(2x).$$

This would require (matching coefficients of sines and cosines)

$$-4A = 20 \quad \text{and} \quad -2A = 0.$$

**This is impossible as it would require $-5 = 0$!**

# General Form: sines and cosines

We must think of our equation $y'' - y' = 20\sin(2x)$ as

$$y'' - y' = 20\sin(2x) + 0\cos(2x).$$

The correct format for $y_p$ is

$$y_p = A\sin(2x) + B\cos(2x).$$

# Examples of Forms of $y_p$ based on $g$ (Trial Guesses)

(a) $g(x) = 1$ (or really any constant)

$y_p = A$ a constant

(b) $g(x) = x - 7$

$y_p = Ax + B$ 1st degree poly.

# Examples of Forms of $y_p$ based on $g$ (Trial Guesses)

(c) $g(x) = 5x^2$

$y_p = Ax^2 + Bx + C$ 2nd degree poly.

(d) $g(x) = 3x^3 - 5$

$y_p = Ax^3 + Bx^2 + Cx + D$ 3rd degree poly.

# Examples of Forms of $y_p$ based on $g$ (Trial Guesses)

(e) $g(x) = xe^{3x}$

$y_p = (Ax+B)e^{3x}$

1st degree poly. times $e^{3x}$

(f) $g(x) = \cos(7x)$

Linear combo of

$y_p = A\cos(7x) + B\sin(7x)$

$\cos(7x)$ and $\sin(7x)$

# Examples of Forms of $y_p$ based on $g$ (Trial Guesses)

(g) $g(x) = \sin(2x) - \cos(4x)$

$$y_p = A\sin(2x) + B\cos(2x) + C\cos(4x) + D\sin(4x)$$

(h) $g(x) = x^2\sin(3x)$

$$y_p = (Ax^2 + Bx + C)\sin(3x) + (Dx^2 + Ex + F)\cos(3x)$$

$2^{nd}$ degree poly. time lin. combo of $\sin(3x)$ and $\cos(3x)$

# Examples of Forms of $y_p$ based on $g$ (Trial Guesses)

(i) $g(x) = e^x \cos(2x)$

$$y_p = Ae^{x}\cos(2x) + Be^{x}\sin(2x)$$

$e^{x}$ times a linear combo of $\sin(2x)$ and $\cos(2x)$.

(j) $g(x) = xe^{-x}\sin(\pi x)$

$$y_p = (Ax+B)e^{-x}\sin(\pi x) + (Cx+D)e^{-x}\cos(\pi x)$$

# The Superposition Principle

**Example:** Determine the correct form of the particular solution using the method of undetermined coefficients for the ODE

$$y'' - 4y' + 4y = \underbrace{6e^{-3x}}_{g_1(x)} + \underbrace{16x^2}_{g_2(x)}$$

For $g_1$: $y_{p_1} = Ae^{-3x}$ $\quad$ $y_{p_1}$ solves $y'' - 4y' + 4y = 6e^{-3x}$

For $g_2$: $y_{p_2} = Bx^2 + Cx + D$ $\quad$ and $y_{p_2}$ solves $y'' - 4y' + 4y = 16x^2$

By the principle of superposition, the form of $y_p$ is

$$y_p = y_{p_1} + y_{p_2} = Ae^{-3x} + Bx^2 + Cx + D$$

From our examples, we found

$$y_{p_1} = \frac{6}{25} e^{-3x} \quad \text{and}$$

$$y_{p_2} = 4x^2 + 8x + 6$$

So a particular solution to

$$y'' - 4y' + 4y = 6e^{-3x} + 16x^2 \quad \text{is}$$

$$\boxed{y_p = \frac{6}{25} e^{-3x} + 4x^2 + 8x + 6}$$