# September 28 Math 3260 sec. 57 Fall 2017

## Section 4.1: Vector Spaces and Subspaces

**Definition** A *real* **vector space** is a nonempty set $V$ of objects called *vectors* together with two operations called *vector addition* and *scalar multiplication* such that: For all $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $V$, and for any scalars $c$ and $d$

1. The sum $\mathbf{u}+\mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$ is in $V$.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. There exists a **zero** vector $\mathbf{0}$ in $V$ such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each vector $\mathbf{u}$ there exists a vector $-\mathbf{u}$ such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. For each scalar $c$, $c\mathbf{u}$ is in $V$.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=d(c\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$

# Remarks

- $V$ is more accurately called a *real vector space* when we assume that the relevant scalars are the real numbers.
- Property 1. is that $V$ is **closed** under (a.k.a. with respect to) vector addition.
- Property 6. is that $V$ is **closed** under scalar multiplication.
- A vector space has the same basic *structure* as $\mathbb{R}^n$
- These are **axioms.** We assume (not ”prove”) that they hold for vector space $V$. However, they can be used to **prove or disprove** that a given set (with operations) is actually a vector space.

# Examples of Vector Spaces

(1) $\mathbb{P}_n$, the set of all polynomials with real coefficients of degree at most $n$

$$\mathbb{P}_n = \{\mathbf{p}(t) = p_0 + p_1 t + \cdots + p_n t^n \mid p_0, p_1, \ldots, p_n \in \mathbb{R}\}, \text{ where}$$

$$(\mathbf{p}+\mathbf{q})(t) = \mathbf{p}(t) + \mathbf{q}(t) = (p_0+q_0) + (p_1+q_1)t + \cdots + (p_n+q_n)t^n,$$

$$(c\mathbf{p})(t) = c\mathbf{p}(t) = cp_0 + cp_1 t + \cdots + cp_n t^n.$$

(2) Let $V$ be the set of all differentiable, real valued functions $f(x)$ defined for $-\infty < x < \infty$ with the property that $f(0) = 0$. Vector addition and scalar multiplication in the standard way for functions—i.e.

$$(f+g)(x) = f(x) + g(x), \quad \text{and} \quad (cf)(x) = cf(x).$$

# A set that is not a Vector Space

Let $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, \mid x \leq 0, y \leq 0 \right\}$ with regular vector addition and scalar multiplication in $\mathbb{R}^2$. Note $V$ is the third quadrant in the $xy$-plane.

We saw that this is not a vector space. One reason is that even if $\mathbf{u}$ is in $V$, $c\mathbf{u}$ is not necessarily in $V$. In fact, if $\mathbf{u} \neq \mathbf{0}$, then $c\mathbf{u}$ is not in $V$ for any negative number $c$.

# Theorem

Let $V$ be a vector space. For each $\mathbf{u}$ in $V$ and scalar $c$

$$\begin{aligned} 0\mathbf{u} &= \mathbf{0} \\ c\mathbf{0} &= \mathbf{0} \\ -1\mathbf{u} &= -\mathbf{u} \end{aligned}$$

# Subspaces

**Definition:** A **subspace** of a vector space $V$ is a subset $H$ of $V$ for which

a) The zero vector is in $H$[1]

b) $H$ is closed under vector addition. (i.e. $\mathbf{u}, \mathbf{v}$ in $H$ implies $\mathbf{u} + \mathbf{v}$ is in $H$)

c) $H$ is closed under scalar multiplication. (i.e. $\mathbf{u}$ in $H$ implies $c\mathbf{u}$ is in $H$)

[1] This is sometimes replaced with the condition that $H$ is nonempty.

# Example

Consider $\mathbb{R}^n$ and let $V$ =Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ be a nonempty ($p \geq 1$) subset of $\mathbb{R}^n$ . Show that $V$ is a subspace.

Note $\vec{0} = 0\vec{v}_1 + 0\vec{v}_2 + \cdots + 0\vec{v}_p \Rightarrow \vec{0}$ is in $V$.

If $\vec{u}$ and $\vec{w}$ are in $V$

$\vec{u} = c_1\vec{v}_1 + \cdots + c_p\vec{v}_p$ and $\vec{w} = d_1\vec{v}_1 + \cdots + d_p\vec{v}_p$

Then $\vec{u} + \vec{w} = (c_1+d_1)\vec{v}_1 + (c_2+d_2)\vec{v}_2 + \cdots + (c_p+d_p)\vec{v}_p$

So $\vec{u} + \vec{w}$ is in $V$.

Also $k\vec{u} = kc_1\vec{v}_1 + kc_2\vec{v}_2 + \cdots + kc_p\vec{v}_p$. for scalar $k$

So $k\vec{u}$ is also in $V$.

So $V = \text{Span}\{\vec{v}_1, \ldots, \vec{v}_p\}$ has the zero vector in it, and it's closed under both operations.

Hence $V$ is a subspace of $\mathbb{R}^n$.

# Example

Determine which of the following is a subspace of $\mathbb{R}^2$.

(a) The set of all vectors of the form $\mathbf{u} = (u_1, 0)$.

This is a subspace. In fact, this is $\text{Span}\{\vec{e}_1\}$. By the previous exercise, this is a subspace.

# Example continued

(b) The set of all vectors of the form $\mathbf{u} = (u_1, 1)$.

This is not a subspace of $\mathbb{R}^2$.

$\vec{0}$ is not in it.

It also fails to be closed under either operation.

# Definition: Linear Combination and Span

**Definition** Let $V$ be a vector space and $\mathbf{v}_1$, $\mathbf{v}_2, \ldots, \mathbf{v}_p$ be a collection of vectors in $V$. A **linear combination** of the vectors is a vector $\mathbf{u}$

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

for some scalars $c_1, c_2, \ldots, c_p$.

**Definition** The **span**, $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$, is the subet of $V$ consisting of all linear combinations of the vectors $\mathbf{v}_1$, $\mathbf{v}_2, \ldots, \mathbf{v}_p$.

# Theorem

**Theorem:** If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$, for $p \geq 1$, are vectors in a vector space $V$, then Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$, is a subspace of $V$.

**Remark** This is called the **subspace of $V$ spanned by (or generated by)** $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. Moreover, if $H$ is any subspace of $V$, a **spanning set** for $H$ is any set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ such that $H =$Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

## Example

$M^{2\times 2}$ denotes the set of all $2\times 2$ matrices with real entries. Consider the subset $H$ of $M^{2\times 2}$

$$H=\left\{\begin{bmatrix} a & 0\\ 0 & b\end{bmatrix}\ \middle|\ a,\, b\in\mathbb{R}\right\}.$$

Show that $H$ is a subspace of $M^{2\times 2}$ by finding a spanning set. That is, show that $H=$Span$\{\mathbf{v}_1,\mathbf{v}_2\}$ for some appropriate vectors.

Let $\vec{u}$ in $H$ be $\vec{u}=\begin{bmatrix} a & 0\\ 0 & b\end{bmatrix}$. Then

$$\vec{u}=\begin{bmatrix} a & 0\\ 0 & b\end{bmatrix}=a\begin{bmatrix} 1 & 0\\ 0 & 0\end{bmatrix}+b\begin{bmatrix} 0 & 0\\ 0 & 1\end{bmatrix}$$

Let $\vec{v}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Then $H = \text{Span}\{\vec{v}_1, \vec{v}_2\}$ so $H$ is a subspace of $M^{2\times 2}$.

# Section 4.2: Null & Column Spaces, Linear Transformations

**Definition:** Let $A$ be an $m \times n$ matrix. The **null space** of $A$, denoted[2] by Nul $A$, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. That is

$$\text{Nul}\, A = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}.$$

We can say that Nul $A$ is the subset of $\mathbb{R}^n$ that gets mapped to the zero vector under the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$.

[2] Some authors will write Null($A$)—I tend to write two ells.

# Example

Determine Nul $A$ where

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 7 \end{bmatrix}.$$

$A\vec{x} = \vec{0}$

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 1 & 2 & 7 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$x_1 = -3x_3$

$x_2 = -2x_3$

$x_3$ - free

$$\vec{x} = \begin{bmatrix} -3x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$$

$$\text{Nul}\, A = \text{Span}\left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

# Theorem

For $m \times n$ matrix $A$, Nul $A$ is a subspace of $\mathbb{R}^n$.

$\vec{0}$ always satisfies $A\vec{x} = \vec{0}$.

$\vec{x}$ must be in $\mathbb{R}^n$ for $A\vec{x}$ to be defined.

Nul A is closed under the operations.

See exam 1 problem #4.

# Example

For a given matrix, a spanning set for Nul$A$ gives an *explicit* description of this subspace. Find a spanning set for Nul $A$ where

$$A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 1 & 2 & 6 & -5 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 0 & 2 & -1 \\ 1 & 2 & 6 & -5 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 2 & -2 \end{bmatrix}$$

$x_1 = -2x_3 + x_4$

$x_2 = -2x_3 + 2x_4$

$x_3, x_4$ - free

$$\vec{x} = \begin{bmatrix} -2x_3 + x_4 \\ -2x_3 + 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Nul } A = \text{Span}\left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$