Section 4.1: Vector Spaces and Subspaces

Definition A real vector space is a nonempty set $V$ of objects called vectors together with two operations called vector addition and scalar multiplication such that: For all $u, v,$ and $w$ in $V,$ and for any scalars $c$ and $d$

1. The sum $u + v$ of $u$ and $v$ is in $V.$
2. $u + v = v + u.$
3. $(u + v) + w = u + (v + w).$
4. There exists a zero vector $0$ in $V$ such that $u + 0 = u.$
5. For each vector $u$ there exists a vector $-u$ such that $u + (-u) = 0.$
6. For each scalar $c$, $cu$ is in $V.$
7. $c(u + v) = cu + cv.$
8. $(c + d)u = cu + du.$
9. $c(du) = d(au) = (cd)u.$
10. $1u = u$
Remarks

- $V$ is more accurately called a *real vector space* when we assume that the relevant scalars are the real numbers.

- Property 1. is that $V$ is **closed** under (a.k.a. with respect to) vector addition.

- Property 6. is that $V$ is **closed** under scalar multiplication.

- A vector space has the same basic *structure* as $\mathbb{R}^n$.

- These are **axioms**. We assume (not ”prove”) that they hold for vector space $V$. However, they can be used to **prove or disprove** that a given set (with operations) is actually a vector space.
Examples of Vector Spaces

(1) $\mathbb{P}_n$, the set of all polynomials with real coefficients of degree at most $n$

$$\mathbb{P}_n = \{ p(t) = p_0 + p_1 t + \cdots + p_n t^n \mid p_0, p_1, \ldots, p_n \in \mathbb{R} \},$$

where

$$(p + q)(t) = p(t) + q(t) = (p_0 + q_0) + (p_1 + q_1)t + \cdots + (p_n + q_n)t^n,$$

$$(cp)(t) = cp(t) = cp_0 + cp_1 t + \cdots + cp_n t^n.$$

(2) Let $V$ be the set of all differentiable, real valued functions $f(x)$ defined for $-\infty < x < \infty$ with the property that $f(0) = 0$. Vector addition and scalar multiplication in the standard way for functions—i.e.

$$(f + g)(x) = f(x) + g(x), \quad \text{and} \quad (cf)(x) = cf(x).$$
A set that is not a Vector Space

Let \( V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, \; |x \leq 0, y \leq 0 \right\} \) with regular vector addition and scalar multiplication in \( \mathbb{R}^2 \). Note \( V \) is the third quadrant in the \( xy \)-plane.

We saw that this is not a vector space. One reason is that even if \( u \) is in \( V \), \( cu \) is not necessarily in \( V \). In fact, if \( u \neq 0 \), then \( cu \) is not in \( V \) for any negative number \( c \).
Theorem

Let $V$ be a vector space. For each $u$ in $V$ and scalar $c$

\[
0u = 0
\]

\[
c0 = 0
\]

\[-1u = -u\]
Subspaces

Definition: A **subspace** of a vector space $V$ is a subset $H$ of $V$ for which

a) The zero vector is in $H$\(^1\)

b) $H$ is closed under vector addition. (i.e. $u, v$ in $H$ implies $u + v$ is in $H$)

c) $H$ is closed under scalar multiplication. (i.e. $u$ in $H$ implies $cu$ is in $H$)

\(^1\)This is sometimes replaced with the condition that $H$ is nonempty.
Example

Consider \( \mathbb{R}^n \) and let \( V = \text{Span}\{v_1, v_2, \ldots, v_p\} \) be a nonempty \((p \geq 1)\) subset of \( \mathbb{R}^n \). Show that \( V \) is a subspace.

\[ \vec{0} = 0\vec{v}_1 + 0\vec{v}_2 + \ldots + 0\vec{v}_p \quad \text{so} \quad \vec{0} \text{ is in } V. \]

Let \( \vec{u} \) and \( \vec{w} \) be in \( V \). Then

\[ \vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_p\vec{v}_p \quad \text{and} \quad \vec{w} = d_1\vec{v}_1 + d_2\vec{v}_2 + \ldots + d_p\vec{v}_p \]

for some scalars \( c_1, \ldots, c_p \) and \( d_1, \ldots, d_p \)

\[ \vec{u} + \vec{w} = (c_1 + d_1)\vec{v}_1 + (c_2 + d_2)\vec{v}_2 + \ldots + (c_p + d_p)\vec{v}_p \]

which is in \( V \).
Similarly, for any scalar $k$

$$k\vec{u} = k\vec{v}_1 + k\vec{v}_2 + \ldots + k\vec{v}_p$$

which is in $V$.

$V$ contains $\vec{0}$ and is closed under both operations. $V$ is a subspace of $\mathbb{R}^n$. 
Example
Determine which of the following is a subspace of $\mathbb{R}^2$.

(a) The set of all vectors of the form $\mathbf{u} = (u_1, 0)$.

Note $\mathbf{u} = u_1(1, 0)$. Hence this set is $\text{Span} \{ e_1 \}$. In light of the previous example, this is a subspace of $\mathbb{R}^2$. 
Example continued

(b) The set of all vectors of the form $\mathbf{u} = (u_1, 1)$.

This is not a subspace. It's not closed under either operation nor is $\mathbf{0}$ in it.
Definition: Linear Combination and Span

**Definition** Let $V$ be a vector space and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ be a collection of vectors in $V$. A **linear combination** of the vectors is a vector $\mathbf{u}$

$$\mathbf{u} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_p \mathbf{v}_p$$

for some scalars $c_1, c_2, \ldots, c_p$.

**Definition** The **span**, $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$, is the subset of $V$ consisting of all linear combinations of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$. 
**Theorem**

**Theorem:** If \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p \), for \( p \geq 1 \), are vectors in a vector space \( V \), then \( \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\} \), is a subspace of \( V \).

**Remark** This is called the **subspace of** \( V \) spanned by (or generated by) \( \{\mathbf{v}_1, \ldots, \mathbf{v}_p\} \). Moreover, if \( H \) is any subspace of \( V \), a **spanning set for** \( H \) is any set of vectors \( \{\mathbf{v}_1, \ldots, \mathbf{v}_p\} \) such that \( H = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} \).
Example

$M^{2 \times 2}$ denotes the set of all $2 \times 2$ matrices with real entries. Consider the subset $H$ of $M^{2 \times 2}$

$$H = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}.$$ 

Show that $H$ is a subspace of $M^{2 \times 2}$ by finding a spanning set. That is, show that $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ for some appropriate vectors.

For $\mathbf{u}$ in $H$, $\mathbf{u} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ for some $a, b$ in $\mathbb{R}$.

Note

$$\mathbf{u} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
we can take $\vec{v}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Then $\mathcal{H} = \text{Span} \{ \vec{v}_1, \vec{v}_2 \}$. 
Section 4.2: Null & Column Spaces, Linear Transformations

Definition: Let $A$ be an $m \times n$ matrix. The null space of $A$, denoted by Nul $A$, is the set of all solutions of the homogeneous equation $Ax = 0$. That is

$$\text{Nul } A = \{ x \in \mathbb{R}^n \mid Ax = 0 \}.$$ 

We can say that Nul $A$ is the subset of $\mathbb{R}^n$ that gets mapped to the zero vector under the linear transformation $x \mapsto Ax$.

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$^2$Some authors will write Null$(A)$—I tend to write two ells.
Example

Determine Nul $A$ where

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 7 \end{bmatrix}.$$  

We consider $A\vec{x} = \vec{0}$

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 1 & 2 & 7 & 0 \end{bmatrix} \xrightarrow{\text{ref}} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

We can say that Nul $A$ is the set of all $\vec{x}$ in $\mathbb{R}^3$ of the form

$$\vec{x} = \begin{bmatrix} -3x^3 \\ -2x^3 \\ x^3 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}.$$  Note Nul $A = \text{Span} \left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} \right\}.$
Theorem

For $m \times n$ matrix $A$, Nul $A$ is a subspace of $\mathbb{R}^n$.

For $A\vec{x}$ defined $\vec{x}$ must be in $\mathbb{R}^n$.

Since $\vec{0}$ always solves $A\vec{x} = \vec{0}$, $\vec{0}$ is in Nul $A$.

Nul $A$ is also closed under both operations, see for example exam 1 problem #4.
Example

For a given matrix, a spanning set for Nul\(A\) gives an *explicit* description of this subspace. Find a spanning set for Nul \(A\) where

\[
A = \begin{bmatrix}
1 & 0 & 2 & -1 \\
1 & 2 & 6 & -5
\end{bmatrix}.
\]

\[
\begin{bmatrix}
1 & 0 & 2 & -1 \\
1 & 2 & 6 & -5
\end{bmatrix} \xrightarrow{\text{ref}} \begin{bmatrix}
1 & 0 & 2 & -1 \\
0 & 1 & 2 & -2
\end{bmatrix}
\]

\(x_1 = -2x_3 + x_4\)

\(x_2 = -2x_3 + 2x_4\)

\(x_3, x_4 - \text{free}\)

\(\tilde{x} \text{ in Nul} A\) looks like

\[
\tilde{x} = \begin{bmatrix}
-2x_3 + x_4 \\
-2x_3 + 2x_4 \\
x_3 \\
x_4
\end{bmatrix} = x_3 \begin{bmatrix}
-2 \\
-2 \\
1 \\
1
\end{bmatrix} + x_4 \begin{bmatrix}
1 \\
2 \\
0 \\
1
\end{bmatrix}
\]
So

\[ \text{Null } A = \text{ Span } \left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \]