September 28 Math 3260 sec. 58 Fall 2017

Section 4.1: Vector Spaces and Subspaces

Definition A *real* **vector space** is a nonempty set V of objects called *vectors* together with two operations called *vector addition* and *scalar multiplication* such that: For all $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V, and for any scalars c and d

1. The sum $\mathbf{u}+\mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$ is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. There exists a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each vector $\mathbf{u}$ there exists a vector $-\mathbf{u}$ such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. For each scalar c, $c\mathbf{u}$ is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=d(c\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$

Remarks

- V is more accurately called a *real vector space* when we assume that the relevant scalars are the real numbers.
- Property 1. is that V is **closed** under (a.k.a. with respect to) vector addition.
- Property 6. is that V is **closed** under scalar multiplication.
- A vector space has the same basic *structure* as $\mathbb{R}^n$
- These are **axioms.** We assume (not ”prove”) that they hold for vector space V. However, they can be used to **prove or disprove** that a given set (with operations) is actually a vector space.

Examples of Vector Spaces

(1) $\mathbb{P}_n$, the set of all polynomials with real coefficients of degree at most n

$$\mathbb{P}_n = \{\mathbf{p}(t) = p_0 + p_1 t + \cdots + p_n t^n \mid p_0, p_1, \dots, p_n \in \mathbb{R}\}, \text{ where}$$

$$(\mathbf{p}+\mathbf{q})(t) = \mathbf{p}(t) + \mathbf{q}(t) = (p_0+q_0) + (p_1+q_1)t + \cdots + (p_n+q_n)t^n,$$

$$(c\mathbf{p})(t) = c\mathbf{p}(t) = cp_0 + cp_1 t + \cdots + cp_n t^n.$$

(2) Let V be the set of all differentiable, real valued functions $f(x)$ defined for $-\infty < x < \infty$ with the property that $f(0) = 0$. Vector addition and scalar multiplication in the standard way for functions—i.e.

$$(f+g)(x) = f(x) + g(x), \quad \text{and} \quad (cf)(x) = cf(x).$$

A set that is not a Vector Space

Let $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, \mid x \leq 0, y \leq 0 \right\}$ with regular vector addition and scalar multiplication in $\mathbb{R}^2$. Note V is the third quadrant in the xy-plane.

We saw that this is not a vector space. One reason is that even if $\mathbf{u}$ is in V, $c\mathbf{u}$ is not necessarily in V. In fact, if $\mathbf{u} \neq \mathbf{0}$, then $c\mathbf{u}$ is not in V for any negative number c.

Theorem

Let V be a vector space. For each $\mathbf{u}$ in V and scalar c

$$\begin{aligned} 0\mathbf{u} &= \mathbf{0} \\ c\mathbf{0} &= \mathbf{0} \\ -1\mathbf{u} &= -\mathbf{u} \end{aligned}$$

Subspaces

Definition: A **subspace** of a vector space V is a subset H of V for which

a) The zero vector is in H[1]

b) H is closed under vector addition. (i.e. $\mathbf{u}, \mathbf{v}$ in H implies $\mathbf{u} + \mathbf{v}$ is in H)

c) H is closed under scalar multiplication. (i.e. $\mathbf{u}$ in H implies $c\mathbf{u}$ is in H)

[1] This is sometimes replaced with the condition that H is nonempty.

Example

Consider $\mathbb{R}^n$ and let $V = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ be a nonempty ($p \geq 1$) subset of $\mathbb{R}^n$. Show that V is a subspace.

$\vec{0} = 0\vec{v}_1 + 0\vec{v}_2 + \cdots + 0\vec{v}_p$ so $\vec{0}$ is in V.

Let $\vec{u}$ and $\vec{w}$ be in V. Then

$\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p$ and $\vec{w} = d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_p\vec{v}_p$

for some scalars $c_1, \ldots, c_p$ and $d_1, \ldots, d_p$

$\vec{u} + \vec{w} = (c_1 + d_1)\vec{v}_1 + (c_2 + d_2)\vec{v}_2 + \cdots + (c_p + d_p)\vec{v}_p$

which is in V.

Similarly, for any scalar k

$$k\vec{u} = kc_1\vec{v}_1 + kc_2\vec{v}_2 + \cdots + kc_p\vec{v}_p$$

which is in V.

V contains $\vec{0}$ and is closed under both operations. V is a subspace of $\mathbb{R}^n$.

Example

Determine which of the following is a subspace of $\mathbb{R}^2$.

(a) The set of all vectors of the form $\mathbf{u} = (u_1, 0)$.

Note $\vec{u} = u_1(1,0)$. Hence this set is $\text{Span}\{\vec{e}_1\}$. In light of the previous example, this is a subspace of $\mathbb{R}^2$.

Example continued

(b) The set of all vectors of the form $\mathbf{u} = (u_1, 1)$.

This is not a subspace. It's not closed under either operation nor is $\vec{0}$ in it.

Definition: Linear Combination and Span

Definition Let V be a vector space and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ be a collection of vectors in V. A **linear combination** of the vectors is a vector $\mathbf{u}$

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

for some scalars $c_1, c_2, \ldots, c_p$.

Definition The **span**, $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$, is the subet of V consisting of all linear combinations of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$.

Theorem

Theorem: If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$, for $p \geq 1$, are vectors in a vector space V, then Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$, is a subspace of V.

Remark This is called the **subspace of V spanned by (or generated by)** $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. Moreover, if H is any subspace of V, a **spanning set** for H is any set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ such that $H =$Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

Example

$M^{2\times 2}$ denotes the set of all 2×2 matrices with real entries. Consider the subset H of $M^{2\times 2}$

$$H = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}.$$

Show that H is a subspace of $M^{2\times 2}$ by finding a spanning set. That is, show that H =Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ for some appropriate vectors.

For $\vec{u}$ in H $\quad \vec{u} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ for some a, b in $\mathbb{R}$.

Note

$$\vec{u} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

We can take $\vec{v}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Then $H = \mathrm{Span}\{\vec{v}_1, \vec{v}_2\}$.

Section 4.2: Null & Column Spaces, Linear Transformations

Definition: Let A be an $m \times n$ matrix. The **null space** of A, denoted[2] by Nul A, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. That is

$$\text{Nul}\, A = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}.$$

We can say that Nul A is the subset of $\mathbb{R}^n$ that gets mapped to the zero vector under the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$.

[2] Some authors will write Null(A)—I tend to write two ells.

Example

Determine Nul A where

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 7 \end{bmatrix}.$$

We consider

$A\vec{x} = \vec{0}$

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 1 & 2 & 7 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

$x_1 = -3x_3$

$x_2 = -2x_3$

x_3 - free

We can say that Nul A is the set of all $\vec{x}$ in $\mathbb{R}^3$ of the form

$$\vec{x} = \begin{bmatrix} -3x^3 \\ -2x^3 \\ x^3 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}.$$

Note Nul $A = \text{Span}\left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} \right\}$.

Theorem

For $m \times n$ matrix A, Nul A is a subspace of $\mathbb{R}^n$.

For $A\vec{x}$ defined $\vec{x}$ must be in $\mathbb{R}^n$.

Since $\vec{0}$ always solves $A\vec{x} = \vec{0}$, $\vec{0}$ is in Nul A

Nul A is also closed under both operations,

see for example exam 1 problem #4.

Example

For a given matrix, a spanning set for Nul$A$ gives an *explicit* description of this subspace. Find a spanning set for Nul A where

$$A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 1 & 2 & 6 & -5 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 0 & 2 & -1 \\ 1 & 2 & 6 & -5 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 2 & -2 \end{bmatrix}$$

$x_1 = -2x_3 + x_4$

$x_2 = -2x_3 + 2x_4$

x_3, x_4 – free

$\vec{x}$ in Nul$A$ looks like

$$\vec{x} = \begin{bmatrix} -2x_3 + x_4 \\ -2x_3 + 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

so

$$\text{Nul}\, A = \text{Span}\left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$