Sept. 2 Math 1190 sec. 51 Fall 2016

Section 1.4: Limits and Continuity of Trigonometric, Exponential and Logarithmic Functions

Theorem: $\displaystyle \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ and $\displaystyle \lim_{\theta\to 0} \frac{\cos\theta - 1}{\theta} = 0$

The first can be stated in alternative ways

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = \lim_{t\to 0} \frac{\sin t}{t} = \lim_{\heartsuit\to 0} \frac{\sin(3\heartsuit)}{3\heartsuit} = 1$$

The key is that the argument of the sine matches the denominator with these tending to zero.

Example

Evaluate each limit if possible.

$$\lim_{r\to 0} r\csc(2r)$$

$$= \lim_{r\to 0} \frac{r}{\sin(2r)}$$

$$* \lim_{r\to 0} \frac{\sin(2r)}{2r} = 1$$

In general

$$\lim_{\theta\to 0} \frac{\theta}{\sin\theta} = 1$$

$$= \lim_{r\to 0} \frac{r}{\sin(2r)} \cdot \frac{2}{2}$$

$$= \lim_{r\to 0} \frac{1}{2} \frac{2r}{\sin(2r)} = \frac{1}{2} \lim_{r\to 0} \left(\frac{\sin(2r)}{2r}\right)^{-1}$$

$$= \frac{1}{2}(1)^{-1} = \frac{1}{2}$$

Questions

(1) Evaluate if possible $\lim_{x \to 0} \frac{\tan(2x)}{4x}$ $= \lim_{x \to 0} \frac{\frac{\sin(2x)}{\cos(2x)}}{4x}$

$$= \lim_{x \to 0} \frac{1}{2\cos(2x)} \left(\frac{\sin(2x)}{2x} \right)$$

(a) $\frac{1}{4}$

(b) $\frac{1}{2}$

$$= \frac{1}{2 \cdot 1} \cdot 1 = \frac{1}{2}$$

(c) 2

(d) DNE

Question

True/False: Since $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$ we can conclude that

$$\frac{\sin x}{x} = 1$$

False, in fact the equation $\frac{\sin x}{x} = 1$ is never true.

Section 1.5: Infinite Limits, Limits at Infinity, Asymptotes

Here, we will consider limits involving the symbols ∞ (*infinity*) and $-\infty$ (*negative infinity*). They will be used to denote **unboundedness** in the positive and negative directions, respectively.

While ∞ and $-\infty$ are NOT NUMBERS, there is an arithmetic for using these symbols. In particular

- $\infty + \infty = \infty$
- $\infty + c = \infty$ for any real number c
- $\infty \cdot c = \infty$ if $c > 0$ and $\infty \cdot c = -\infty$ if $c < 0$
- $\frac{0}{\infty} = \frac{0}{-\infty} = 0$

Other forms that may appear are indeterminate. The following **are not defined.**

$$\infty - \infty, \quad \frac{\infty}{\infty}, \quad 0 \cdot \infty, \quad \infty^0$$

Infinite Limits

Investigate the limit

$$\lim_{x \to 0} \frac{1}{x^2}$$

x	$f(x) = \frac{1}{x^2}$
-0.1	100
-0.01	10,000
-0.001	1,000,000
0	undefined
0.001	1,000,000
0.01	10,000
0.1	100

$f(x)$ grows without bound as $x \to 0$

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

Infinite Limits

Definition: Let $f(x)$ be defined on an open interval containing c except possibly at c. Then

$$\lim_{x \to c} f(x) = \infty$$

provided $f(x)$ can be made arbitrarily large by taking x sufficiently close to c. (The definition of

$$\lim_{x \to c} f(x) = -\infty$$

is similar except that f can be made arbitrarily large and negative.)

The top limit statement reads *the limit as x approaches c of $f(x)$ equals infinity*.

Infinite Limits

Investigate the limit

$$\lim_{x \to 0} \frac{1}{x}$$

x	$f(x) = \frac{1}{x}$
-0.1	-10
-0.01	-100
-0.001	-1000
0	undefined
0.001	1000
0.01	100
0.1	10

$\lim_{x \to 0} \frac{1}{x}$ DNE

$\Longrightarrow \lim_{x \to 0^-} \frac{1}{x} = -\infty$

$\Longrightarrow \lim_{x \to 0^+} \frac{1}{x} = \infty$

An Observation

Suppose when taking a limit $\lim_{x \to c} f(x)$ we see the form

$$\frac{k}{0} \quad \text{where } k \text{ is any nonzero real number.}$$

Then this limit **MAY** be either ∞ or $-\infty$.

- If we determine that the ratio is positive for all x near c, the limit is ∞.
- If we determine that the ratio is negative for all x near c, the limit is $-\infty$.
- If the ratio can take either sign for x sufficiently close to c, the limit DNE.

Evaluate Each Limit if Possible

(a) $\lim_{x \to 1^-} \frac{2x+1}{x-1}$

$= -\infty$

"The ratio is going to "$\frac{3}{0}$" and is always negative."

$\lim_{x \to 1^-} (2x+1) = 2 \cdot 1 + 1 = 3$

The numerator is going to 3.

$\lim_{x \to 1^-} (x-1) = 0$

$x \to 1^-$ means $x < 1$

$x < 1 \Rightarrow x - 1 < 0$

So $x - 1 \to 0$ through negative numbers.

(b) $\lim_{x \to 1^+} \frac{2x+1}{x-1}$

Again the numerator is tending to 3.

$= \infty$

For $x > 1$, $x - 1 > 0$

The ratio tends to "$\frac{3}{0}$" and is always positive.

(c) $\lim_{x \to 1} \frac{2x+1}{x-1}$ DNE

The one sided limits disagree.

(d) $\lim_{x\to 3} \frac{x - x^2}{|x - 3|}$

$= -\infty$

The ratio tends to " $\frac{-6}{0}$ " and is always negative.

$\lim_{x\to 3} (x - x^2) = 3 - 3^2 = -6$

$\lim_{x\to 3} |x - 3| = 0$

Because of the absolute value bars, $|x-3|$ is positive for all x close to 3.

Question

Evaluate if possible $\lim_{t \to 2} \frac{1}{t-2}$.

(a) ∞

(b) $-\infty$

(c) 4

(d) DNE

$$\lim_{t \to 2^+} \frac{1}{t-2} = \infty$$

and

$$\lim_{t \to 2^-} \frac{1}{t-2} = -\infty$$

Well Known Infinite Limits

Some limits that follow from what we know about these functions

$$\lim_{x\to 0^+} \ln(x) = -\infty$$

$$\lim_{\theta\to \frac{\pi}{2}^-} \tan\theta = \infty \quad \text{and} \quad \lim_{\theta\to \frac{\pi}{2}^+} \tan\theta = -\infty$$

$$\lim_{\theta\to 0^-} \cot\theta = -\infty \quad \text{and} \quad \lim_{\theta\to 0^+} \cot\theta = \infty$$

$$\lim_{\theta\to \frac{\pi}{2}^-} \sec\theta = \infty \quad \text{and} \quad \lim_{\theta\to \frac{\pi}{2}^+} \sec\theta = -\infty$$

$$\lim_{\theta\to 0^-} \csc\theta = -\infty \quad \text{and} \quad \lim_{\theta\to 0^+} \csc\theta = \infty$$

Limits at Infinity

We know what is meant by a limit being infinite (i.e. $f \to \infty$ or $f \to -\infty$). Now, we want to consider limits like

$$\lim_{x\to\infty} f(x) \qquad \text{or like}$$

$$\lim_{x\to-\infty} f(x).$$

What is meant by such a thing, and how is it related to a function's graph?

Definitions

Let f be defined on an interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

provided the value of f can be made arbitrarily close to L by taking x sufficiently large.

Similarly

Defintion: Let f be defined on an interval $(-\infty, a)$. Then

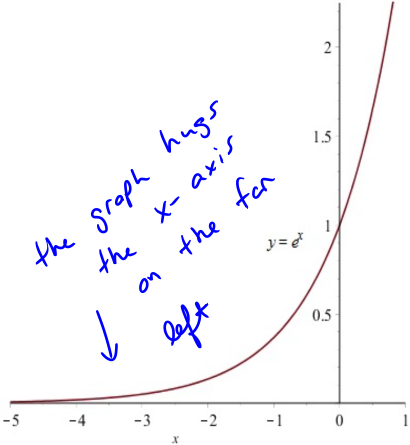
$$\lim_{x \to -\infty} f(x) = L$$

provided the value of f can be made arbitrarily close to L by taking x sufficiently large and negative.

Example

Investigate the limit

$$\lim_{x \to -\infty} e^x = 0$$



the graph hugs the x-axis on the far left

* This is a limit, It is true that $e^x \neq 0$.

Some Results to Remember

Let k be any real number and let p be rational. Then

$$\lim_{x\to\infty} \frac{k}{x^p} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{k}{x^p} = 0$$

The latter holds assuming x^p is defined for $x < 0$.

In essence, if the numerator of a ratio is staying finite while the denominator is becoming infinite, the ratio is tending to zero.

Examples

Evaluate if possible

(a) $\displaystyle\lim_{x\to\infty} \frac{3x^2+2x-1}{x^2+5x+2}$

$$= \lim_{x\to\infty} \left(\frac{3x^2+2x-1}{x^2+5x+2}\right) \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{3+\frac{2}{x}-\frac{1}{x^2}}{1+\frac{5}{x}+\frac{2}{x^2}} = \frac{3+0-0}{1+0+0}$$

$$= \frac{3}{1} = 3$$

we'll use

$$\lim_{x\to\infty} \frac{k}{x^p} = 0$$

Approach:

Identify the highest power of x and multiply and divide by its reciprocal.