September 2 Math 2306 sec 51 Fall 2015

Section 3.1 (1.3, and a peek at 3.2) Applications

A population of dwarf rabbits grows at a rate proportional to the current population. In 2011, there were 58 rabbits. In 2012, the population was up to 89 rabbits. Estimate the number of rabbits expected in the population in 2021.

We let $P(t)$ be the population of rabbits at the time t in years with $t = 0$ in 2011. We then translated the above paragraph into the problem

$$\frac{dP}{dt} = kP, \quad P(0) = 58 \quad \text{and} \quad P(1) = 89.$$

The constant of proportionality k will have to be determined as part of the problem.

We'll solve the IVP

$$\frac{dP}{dt} = kP \qquad P(0) = 58 \qquad\qquad \frac{dP}{dt} - kP = 0$$

$$\frac{1}{P}\frac{dP}{dt} = k \quad \Rightarrow \quad \frac{1}{P}\frac{dP}{dt}\,dt = k\,dt$$

$$\int \frac{1}{P}\,dP = \int k\,dt \qquad\qquad P>0 \text{ as a population}$$

$$\ln P = kt + C \quad \Rightarrow \quad e^{\ln P} = e^{kt+C} = e^{C}e^{kt}$$

$$P = Ae^{kt} \quad \text{where } A = e^{C}$$

$$P(0) = Ae^0 = 58 \Rightarrow A = 58$$

The solution to the IVP is

$$P(t) = 58\, e^{kt}$$

From $P(1) = 89$ $\quad P(1) = 58\, e^{k\cdot 1} = 89$

$$e^k = \frac{89}{58} \Rightarrow k = \ln\left(\frac{89}{58}\right)$$

So $P(t) = 58\, e^{\ln\left(\frac{89}{58}\right)t}$

In 2021, $t = 10$. In 2021, our model predicts a population

$$P(10) = 58\, e^{\ln\left(\frac{89}{58}\right)\cdot 10}$$

$$= 58\, e^{\ln\left(\left(\frac{89}{58}\right)^{10}\right)} = 58\left(\frac{89}{58}\right)^{10}$$

$$\approx 4198$$

Exponential Growth or Decay

If a quantity P changes continuously at a rate proportional to its current level, then it will be governed by a differential equation of the form

$$\frac{dP}{dt} = kP \quad \text{i.e.} \quad \frac{dP}{dt} - kP = 0.$$

Note that this equation is both separable and first order linear. If $k > 0$, P experiences **exponential growth**. If $k < 0$, then P experiences **exponential decay**.

Series Circuits: RC-circuit

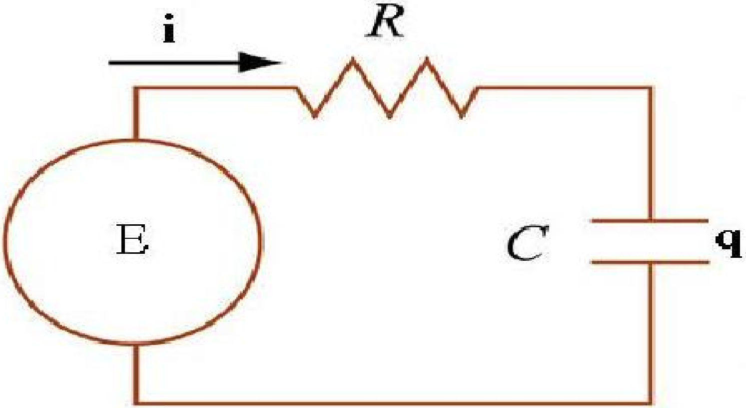


Figure: Series Circuit with Applied Electromotive force E, Resistance R, and Capcitance C. The charge of the capacitor is q and the current $i = \frac{dq}{dt}$.

Series Circuits: LR-circuit

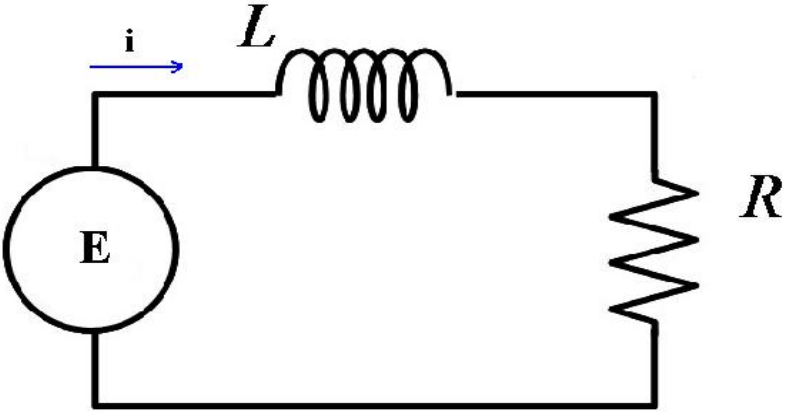


Figure: Series Circuit with Applied Electromotive force E, Inductance L, and Resistance R. The current is i.

Measurable Quantities:

Resistance R in ohms (Ω), Applied voltage E in volts (V),
Inductance L in henries (h), Charge q in coulombs (C),
Capacitance C in farads (f), Current i in amperes (A)

Current is the rate of change of charge with respect to time: $i = \dfrac{dq}{dt}$.

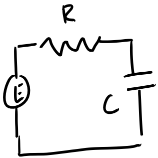
Component	Potential Drop
Inductor	$L\frac{di}{dt}$
Resistor	Ri i.e. $R\frac{dq}{dt}$
Capacitor	$\frac{1}{C}q$

Kirchhoff's Law

The sum of the voltages around a closed circuit is zero.

In other words, the sum of potential drops across the passive components is equal to the applied electromotive force.

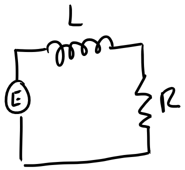
RC circuit



$$E = Ri + \frac{1}{C} q$$

$$R \frac{dq}{dt} + \frac{1}{c} q = E$$

1st order linear eqn.



$$E = L\frac{di}{dt} + Ri$$

another 1^{st} order linear ODE

$$L\frac{di}{dt} + Ri = E$$

Example

A 200 volt battery is applied to an RC series circuit with resistance 1000Ω and capacitance 5×10^{-6} f. Find the charge $q(t)$ on the capacitor if $i(0) = 0.4A$. Determine the charge as $t \to \infty$.

RC circuit $\quad R\frac{dq}{dt} + \frac{1}{C}q = E \quad q'(0) = 0.4$

$$1000\frac{dq}{dt} + \frac{1}{5\cdot 10^{-6}}q = 200$$

$$1000\frac{dq}{dt} + \frac{10^6}{5}q = 200$$

Standard form $\quad \frac{dq}{dt} + \frac{10^6}{5\cdot 10^3}q = \frac{200}{1000}$

$$\frac{dq}{dt} + 200\,q = \frac{1}{5} \qquad q'(0) = 0.4$$

$$P(t) = 200 \Rightarrow \int P(t)\,dt = \int 200\,dt = 200t$$

Integrating factor $\mu = e^{\int P(t)\,dt} = e^{200t}$

$$e^{200t}\frac{dq}{dt} + 200\,e^{200t}q = \frac{1}{5}e^{200t}$$

$$\frac{d}{dt}\left[e^{200t}q\right] = \frac{1}{5}e^{200t}$$

$$\int \frac{d}{dt}\left[e^{200t} q\right] dt = \int \frac{1}{5} e^{200t}\, dt$$

$$e^{200t} q = \frac{1}{5} \cdot \frac{1}{200} e^{200t} + C$$

$$q = \frac{1}{1000} + C e^{-200t}$$

Use $q'(0) = 0.4$

$$q'(t) = -200 C e^{-200t}$$

$$q'(0) = -200 C e^{0} = -200 C = 0.4$$

$$C = \frac{0.4}{-200} = \frac{-4/10}{200} = \frac{-2}{1000} = \frac{-1}{500}$$

Hence $$q(t) = \frac{1}{1000} - \frac{1}{500} e^{-200t}$$

$$\lim_{t \to \infty} q(t) = \lim_{t \to \infty} \left(\frac{1}{1000} - \frac{1}{500} e^{-200t} \right)$$

$$= \frac{1}{1000} + 0$$

$$= \frac{1}{1000}$$

A Classic Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

A Classic Mixing Problem

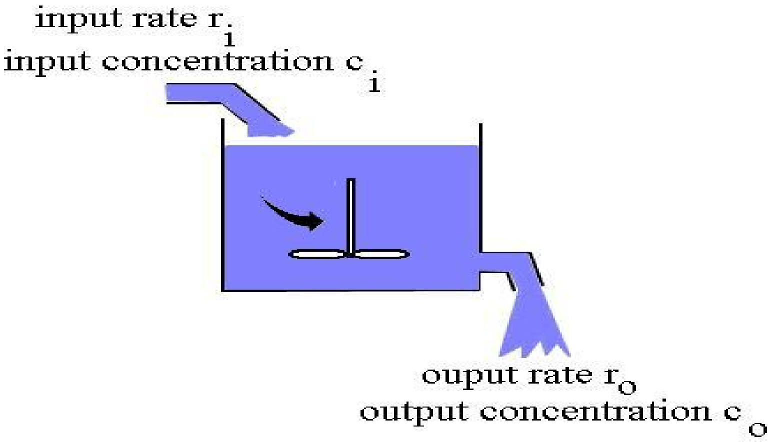


Figure: Spatially uniform composite fluids (e.g. salt & water, gas & ethanol) being mixed. Concentrations of substances change in time.

Building an Equation

A(t) amount of salt @ time t

The rate of change of the amount of salt

$$\frac{dA}{dt} = \left(\begin{array}{c}\textit{input rate}\\ \textit{of salt}\end{array}\right) - \left(\begin{array}{c}\textit{output rate}\\ \textit{of salt}\end{array}\right)$$

The input rate of salt is

$$\text{fluid rate in} \cdot \text{concentration of inflow} \quad = \quad r_i(c_i).$$

The output rate of salt is

$$\text{fluid rate out} \cdot \text{concentration of outflow} \quad = \quad r_o(c_o).$$

Building an Equation

The concentration of the outflowing fluid is

$$\frac{\text{total salt}}{\text{total volume}} = \frac{A(t)}{V(t)} = \frac{A(t)}{V(0) + (r_i - r_o)t}.$$

$$\frac{dA}{dt} = r_i \cdot c_i - r_o \frac{A}{V}.$$

This equation is first order linear.

Solve the Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

$t \sim$ minutes

Incoming fluid

rate : $r_i = 5 \frac{\text{gal}}{\text{min}}$

Conc : $c_i = 2 \frac{\text{lb}}{\text{gal}}$

Outgoing fluid

rate : $r_o = 5 \frac{\text{gal}}{\text{min}}$

Conc : $c_o = \frac{A}{V} = \frac{A \text{ lb}}{500 \text{ gal}}$

$$\frac{dA}{dt} = 5 \cdot 2 \ \frac{\text{lb}}{\text{min}} - 5 \cdot \frac{A}{500} \ \frac{\text{lb}}{\text{min}}$$

$$\frac{dA}{dt} = 10 - \frac{1}{100} A$$

From the first sentence (pure water) $A(0) = 0$

IVP $$\frac{dA}{dt} + \frac{1}{100} A = 10, \quad A(0) = 0$$

We'll solve this on Friday.