Sept. 30 Math 1190 sec. 51 Fall 2016

Section 3.2: Implicit Differentiation; Derivatives of the Inverse Trigonometric Functions

Inverse Functions Suppose $y = f(x)$ and $x = g(y)$ are inverse functions—i.e. $(g \circ f)(x) = g(f(x)) = x$ for all x in the domain of f.

Note: As inverse functions, if

$$f(x_0) = y_0 \quad \text{then} \quad g(y_0) = x_0.$$

This means that

(x_0, y_0) is a point on the graph of f, and

(y_0, x_0) is a point on the graph of g.

Derivatives of Inverse Functions

Theorem: Let f be differentiable on an open interval containing the number x_0. If $f'(x_0) \neq 0$, then g is differentiable at $y_0 = f(x_0)$. Moreover

$$\frac{d}{dy} g(y_0) = g'(y_0) = \frac{1}{f'(x_0)}.$$

Note that this refers to a pair (x_0, y_0) on the graph of f—i.e. (y_0, x_0) on the graph of g. The slope of the curve of f at this point is the reciprocal of the slope of the curve of g at the associated point.

Example

The function $f(x) = x^7 + x + 1$ has an inverse function g. Determine $g'(3)$.

If $y_0 = 3$ and $f(x_0) = y_0 = 3$ then

$$g'(3) = \frac{1}{f'(x_0)}$$

We need to find x_0 so that $f(x_0) = 3$. This requirs $f(x_0) = x_0^7 + x_0 + 1 = 3$

By observation (i.e. clever guessing) $x_0 = 1$.

$$f(x) = x^7 + x + 1 \quad \Rightarrow \quad f'(x) = 7x^6 + 1 + 0 = 7x^6 + 1$$

Then $\quad f'(x_0) = f'(1) = 7(1)^6 + 1 = 7 + 1 = 8$

Finally $\quad g'(3) = \dfrac{1}{f'(1)} = \dfrac{1}{8}$.

Inverse Trigonometric Functions

Recall the definitions of the inverse trigonometric functions.

$$y = \sin^{-1} x \iff x = \sin y, \quad -1 \le x \le 1, \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

quad I + IV

$$y = \cos^{-1} x \iff x = \cos y, \quad -1 \le x \le 1, \quad 0 \le y \le \pi$$

quad I + II

$$y = \tan^{-1} x \iff x = \tan y, \quad -\infty < x < \infty, \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

quad I + IV

Inverse Trigonometric Functions

There are different conventions used for the ranges of the remaining functions. Sullivan and Miranda use

$$y = \cot^{-1} x \quad \Longleftrightarrow \quad x = \cot y, \quad -\infty < x < \infty, \quad 0 < y < \pi$$

I and II

$$y = \csc^{-1} x \quad \Longleftrightarrow \quad x = \csc y, \quad |x| \geq 1, \quad y \in \left(-\pi, -\frac{\pi}{2}\right] \cup \left(0, \frac{\pi}{2}\right]$$

quad III + I

$$y = \sec^{-1} x \quad \Longleftrightarrow \quad x = \sec y, \quad |x| \geq 1, \quad y \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$$

quad I and III

Derivative of the Inverse Sine

Use implicit differentiation to find $\frac{d}{dx}\sin^{-1}x$, and determine the interval over which $y = \sin^{-1}x$ is differentiable.

$y = \sin^{-1}x \implies x = \sin y$ with $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

Take $\frac{d}{dx}$ of this relation.

$$\frac{d}{dx}x = \frac{d}{dx}\sin y$$

chain rule

$$1 = \cos y \frac{dy}{dx}$$

For y such that $\cos y \ne 0$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

Here $g(x) = \sin^{-1} x$

$f(x) = \sin x$

Note that the derivative of g is the reciprocal of the derivative of f.

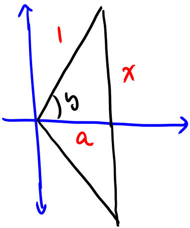
$\cos y = \cos(\sin^{-1} x)$

Since $y = \sin^{-1} x$

Let's find $\cos(\sin^{-1} x)$ as an algebraic expression.

$-\pi/2 \le y \le \frac{\pi}{2}$

$\frac{opp}{hyp} \rightarrow \frac{x}{1} = \sin y$



y in quad I or $\overline{\underline{IV}}$

From the triangle

$a^2 + x^2 = 1^2 \Rightarrow a^2 = 1 - x^2$

$$\cos(\sin^{-1} x) = \frac{\sqrt{1-x^2}}{1} \leftarrow \frac{\text{adj}}{\text{hyp}}$$

$a = \sqrt{1-x^2}$

$$= \sqrt{1-x^2}$$

So

$$\frac{dy}{dx} = \frac{1}{\cos(\sin^{-1} x)} = \frac{1}{\sqrt{1-x^2}}$$

$1 - x^2 > 0$

$x^2 < 1$

$|x| < 1$

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1$$

Examples

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$$

Evaluate each derivative

(a) $\dfrac{d}{dx}\sin^{-1}(e^x)$ $= \dfrac{1}{\sqrt{1-(e^x)^2}} \cdot e^x$

Chain rule

$$= \frac{e^x}{\sqrt{1-e^{2x}}}$$

(b) $\dfrac{d}{dx}\left(\sin^{-1}x\right)^3$

$$= 3\left(\sin^{-1}x\right)^2 \cdot \frac{1}{\sqrt{1-x^2}} = \frac{3(\sin^{-1}x)^2}{\sqrt{1-x^2}}$$

Derivative of the Inverse Tangent

Use implicit differentiation to find $\frac{d}{dx}\tan^{-1} x$, and determine the interval over which $y = \tan^{-1} x$ is differentiable.

$y = \tan^{-1} x \quad \Rightarrow \quad x = \tan y$ with $-\frac{\pi}{2} < y < \frac{\pi}{2}$

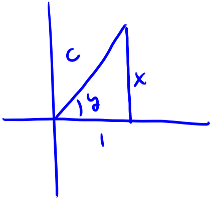
$$\frac{d}{dx} x = \frac{d}{dx} \tan y$$

$$1 = \sec^2 y \cdot \frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sec^2 y}$$

$$\frac{dy}{dx} = \frac{1}{\sec^2(\tan^{-1} x)}$$

$y = \tan^{-1} x$

$\frac{opp}{adj} \to \frac{x}{1} = \tan y$



$c^2 = 1^2 + x^2 \Rightarrow$

$c = \sqrt{1+x^2}$

$\sec y = \frac{\sqrt{1+x^2}}{1}$ $\quad \frac{hyp}{adj}$

$= \sqrt{1+x^2}$

So $\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{(\sqrt{1+x^2})^2} = \frac{1}{1+x^2}$

$$\boxed{\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}} \qquad -\infty < x < \infty$$

An alternative approach:

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

Recall $\tan^2\theta + 1 = \sec^2\theta$, so

$$\sec^2 y = 1 + \tan^2 y \qquad , \quad y = \tan^{-1}x$$

$$= 1 + \tan^2(\tan^{-1}x)$$

$$= 1 + x^2$$

so again $\dfrac{dy}{dx} = \dfrac{1}{1+x^2}$

Questions

$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$

Find $\frac{d^2y}{dx^2}$ where $y = \tan^{-1} x$.

(a) $\frac{d^2y}{dx^2} = \frac{-2x}{(1+x^2)^2}$

(b) $\frac{d^2y}{dx^2} = \left(\frac{1}{1+x^2}\right)^2$

(c) $\frac{d^2y}{dx^2} = \frac{x^2+1-2x}{(1+x^2)^2}$

(d) $\frac{d^2y}{dx^2} = \frac{2x}{(1+x^2)^2}$

Derivative of the Inverse Secant

Theorem: If $f(x) = \sec^{-1} x$, then f is differentiable for all $|x| > 1$ and

$$f'(x) = \frac{d}{dx} \sec^{-1} x = \frac{1}{x\sqrt{x^2 - 1}}.$$

Examples

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$$

Evaluate

(a) $\dfrac{d}{dx}\sec^{-1}(x^2)$ $= \dfrac{1}{x^2\sqrt{(x^2)^2-1}} \cdot (2x)$

$$= \frac{2x}{x^2\sqrt{x^4-1}} = \frac{2}{x\sqrt{x^4-1}}$$

(b) $\dfrac{d}{dx}\tan^{-1}(\sec x)$

$$= \frac{1}{1+(\sec x)^2} \cdot \sec x \tan x = \frac{\sec x \tan x}{1+\sec^2 x}$$

The Remaining Inverse Functions

Due to the trigonometric cofunction identities, it can be shown that

$$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$$

$$\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$$

and

$$\csc^{-1} x = \frac{\pi}{2} - \sec^{-1} x$$

Derivatives of Inverse Trig Functions

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}, \qquad \frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}, \qquad \frac{d}{dx}\cot^{-1}x = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}, \qquad \frac{d}{dx}\csc^{-1}x = -\frac{1}{x\sqrt{x^2-1}}$$