September 30 Math 2306 sec 51 Fall 2015

Section 4.4: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions including polynomials, exponentials, sines/cosines, and sums or products of these.

The method is to assume y_p has the same general *form* as g with undetermined coefficients.

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

Examples of Forms of y_p based on g (Trial Guesses)

(a) $g(x) = 1 \qquad y_p = A$

(b) $g(x) = x - 7 \qquad y_p = Ax + B$

(c) $g(x) = 5x \qquad y_p = Ax + B$

(d) $g(x) = 3x^3 - 5 \qquad y_p = Ax^3 + Bx^2 + Cx + D$

More Trial Guesses

(e) $g(x) = xe^{3x}$ $\quad y_p = (Ax + B)e^{3x}$

(f) $g(x) = \cos(7x)$ $\quad y_p = A\cos(7x) + B\sin(7x)$

(g) $g(x) = \sin(2x) - \cos(4x)$
$y_p = A\sin(2x) + B\cos(2x) + C\sin(4x) + D\cos(4x)$

(h) $g(x) = x^2\sin(3x)$
$y_p = (Ax^2 + Bx + C)\sin(3x) + (Dx^2 + Ex + F)\cos(3x)$

Still More Trial Guesses

(i) $g(x) = e^x \cos(2x)$

$$y_p = Ae^x \cos(2x) + Be^x \sin(2x)$$

(j) $g(x) = x^3 e^{8x}$

$$y_p = (Ax^3 + Bx^2 + Cx + D)e^{8x}$$

(k) $g(x) = xe^{-x} \sin(\pi x)$

$$y_p = (Ax + B)e^{-x} \sin(\pi x) + (Cx + D)e^{-x} \cos(\pi x)$$

The Superposition Principle

Find the general solution of the nonhomogeneous equation

$$y'' - y' = 20\sin(2x) + 4e^{-5x}$$

On 9/28/15, we found that

$$y_p = \frac{2}{15}e^{-5x} \quad \text{is a particular solution of} \quad y'' - y' = 4e^{-5x}$$

and $y_p = -4\sin(2x) + 2\cos(2x)$ is a particular solution of $y'' - y' = 20\sin(2x)$

The principle of superposition tells us that

$$y_p = y_{p_1} + y_{p_2}$$

where $y_{p_1} = \frac{2}{15} e^{-5x}$ and

$$y_{p_2} = -4\sin(2x) + 2\cos(2x)$$

i.e. $y_p = \frac{2}{15} e^{-5x} - 4\sin(2x) + 2\cos(2x)$

To find the general solution, we need y_c.

Solve $y'' - y' = 0$

Characteristic Eqn: $m^2 - m = 0$ $\quad m_1 = 0$

$m(m-1) = 0 \Rightarrow \quad m_2 = 1$

$y_1 = e^{0x} = 1, \quad y_2 = e^{1x} = e^x$

$$y_c = c_1 + c_2 e^x$$

The general solution is

$$\boxed{y = c_1 + c_2 e^x + \frac{2}{15} e^{-5x} - 4\sin(2x) + 2\cos(2x).}$$

A Glitch!

$$y'' - y' = 3e^x$$

$g(x) = 3e^x$, suppose $y_p = Ae^x$

$y_p' = Ae^x$, $y_p'' = Ae^x$

$y_p'' - y_p' = 3e^x$

$Ae^x - Ae^x = 3e^x$

$0 \cdot e^x = 3e^x$ This is false for all possible values of A!

There is a fix: Multiply the guess by x^n where n is the smallest positive integer for which $x^n e^x$ does not solve the homogeneous Eqn.

Here $n=1$ since xe^x does not solve $y''-y'=0$.

Take $y_p = Axe^x$

$$y_p' = Ae^x + Axe^x$$

$$y_p'' = Ae^x + Ae^x + Axe^x = 2Ae^x + Axe^x$$

$$y_p'' - y_p' = 3e^x$$

$$2Ae^x + Axe^x - (Ae^x + Axe^x) = 3e^x$$

$$(A-A)xe^x + (2A-A)e^x = 3e^x$$

$$Ae^x = 3e^x$$

$$A = 3$$

So $y_p = 3xe^x$

The general solution to the DE is

$$y = C_1 + C_2 e^x + 3xe^x.$$

We'll consider cases

Using superposition as needed, begin with assumption:

$$y_p = y_{p_1} + \cdots + y_{p_k}$$

where y_{p_i} has the same **general form** as $g_i(x)$.

Case I: y_p as first written has no part that duplicates the complementary solution y_c. Then this first form will suffice.

Case II: y_p has a term y_{p_i} that duplicates a term in the complementary solution y_c. Multiply that term by x^n, where n is the smallest positive integer that eliminates the duplication.

Case II Examples

Solve the ODE

$$y'' - 2y' + y = -4e^x$$

Get y_c: $\quad y'' - 2y' + y = 0$

Characteristic eqn: $\quad m^2 - 2m + 1 = 0$

$(m-1)^2 = 0 \Rightarrow$ $m = 1$ repeated root

$y_1 = e^x, \quad y_2 = xe^x$

$$y_c = C_1 e^x + C_2 x e^x$$

Get y_p using undetermined coeff.

$g(x) = -4e^x$ First guess $y_p = Ae^x$

This y_p solves the homogeneous eqn.

2nd guess $y_p = Axe^x$, this also solves the homogeneous eqn.

3rd guess $y_p = Ax^2e^x$ this will work.

$$y_p' = 2Axe^x + Ax^2e^x$$

$$y_p'' = 2Ae^x + 2Axe^x + 2Axe^x + Ax^2e^x$$

$$= 2Ae^x + 4Axe^x + Ax^2e^x$$

$$y_p'' - 2y_p' + y_p = -4e^x$$

$$2Ae^x + 4Axe^x + Ax^2e^x - 2(2Axe^x + Ax^2e^x) + Ax^2e^x = -4e^x$$

$$(A - 2A + A)x^2e^x + (4A - 4A)xe^x + 2Ae^x = -4e^x$$

$$2A = -4 \implies A = -2$$

So $\quad y_p = -2x^2e^x$

The general solution to the DE is

$$\boxed{y = C_1e^x + C_2xe^x - 2x^2e^x.}$$

Find the form of the particular soluition

$$y'' - 4y' + 4y = \sin(4x) + xe^{2x}$$

Let $g_1(x) = \sin(4x)$, $g_2(x) = xe^{2x}$

Initial guesses w/o knowledge of y_c would be

$$y_{p_1} = A\sin(4x) + B\cos(4x)$$

and

$$y_{p_2} = (Cx + D)e^{2x}$$

Consider y_c: $y'' - 4y' + 4y = 0$

Characteristic Eqn $\quad m^2 - 4m + 4 = 0$

$$(m-2)^2 = 0 \Rightarrow m = 2$$ repeated

$$y_1 = e^{2x}, \quad y_2 = x e^{2x}$$

y_{p_1} is fine, y_{p_2} is not

$$y_{p_2} = (Cx + D)x^2 e^{2x} = (Cx^3 + Dx^2)e^{2x}$$

$$y_p = A\sin(4x) + B\cos(4x) + (Cx^3 + Dx^2)e^{2x}$$

Find the form of the particular soluition

$$y''' - y'' + y' - y = \cos x + x^4$$

$g_1 = \cos x \,, \qquad g_2 = x^4$

First guess $\qquad y_{p_1} = A\cos x + B\sin x$

$$y_{p_2} = Cx^4 + Dx^3 + Ex^2 + Fx + G$$

Finding y_c: $\qquad m^3 - m^2 + m - 1 = 0 \qquad m_1 = 1$

$$m^2(m-1) + m - 1 = 0 \qquad m_{2,3} = \pm i$$

$$(m-1)(m^2+1) = 0 \qquad = 0 \pm 1i$$

$y_1 = e^x$, $y_2 = \cos x$, $y_3 = \sin x$

Modify y_{p_1}

$$y_{p_1} = (A\cos x + B\sin x)\,x$$

$$= Ax\cos x + Bx\sin x$$

this will work

$$y_p = Ax\cos x + Bx\sin x + Cx^4 + Dx^3 + Ex^2 + Fx + G$$