

Section 4.4: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions including polynomials, exponentials, sines/cosines, and sums or products of these.

The method is to assume y_p has the same general form as g with undetermined coefficients.

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

Examples of Forms of y_p based on g (Trial Guesses)

(a) $g(x) = 1 \quad y_p = A$

(b) $g(x) = x - 7 \quad y_p = Ax + B$

(c) $g(x) = 5x \quad y_p = Ax + B$

(d) $g(x) = 3x^3 - 5 \quad y_p = Ax^3 + Bx^2 + Cx + D$

More Trial Guesses

(e) $g(x) = xe^{3x}$ $y_p = (Ax + B)e^{3x}$

(f) $g(x) = \cos(7x)$ $y_p = A\cos(7x) + B\sin(7x)$

(g) $g(x) = \sin(2x) - \cos(4x)$
 $y_p = A\sin(2x) + B\cos(2x) + C\sin(4x) + D\cos(4x)$

(h) $g(x) = x^2 \sin(3x)$
 $y_p = (Ax^2 + Bx + C)\sin(3x) + (Dx^2 + Ex + F)\cos(3x)$

Still More Trial Guesses

$$(i) g(x) = e^x \cos(2x)$$

$$y_p = A e^x \cos(2x) + B e^x \sin(2x)$$

$$(j) g(x) = x^3 e^{8x}$$

$$y_p = (Ax^3 + Bx^2 + Cx + D) e^{8x}$$

$$(k) g(x) = xe^{-x} \sin(\pi x)$$

$$y_p = (Ax + B)e^{-x} \sin(\pi x) + (Cx + D)e^{-x} \cos(\pi x)$$

The Superposition Principle

Find the general solution of the nonhomogeneous equation

$$y'' - y' = 20 \sin(2x) + 4e^{-5x}$$

On 9/28/15, we found that

$$y_p = \frac{2}{15} e^{-5x} \text{ is a particular solution of } y'' - y' = 4e^{-5x}$$

and $y_p = -4 \sin(2x) + 2 \cos(2x)$ is a particular solution of
 $y'' - y' = 20 \sin(2x)$

By super position,

$$y_p = \frac{2}{15} e^{-5x} - 4 \sin(2x) + 2 \cos(2x).$$

We still need y_c . Solve

$$y'' - y' = 0$$

Characteristic Eqn $m^2 - m = 0$ $m_1 = 0$
 $m(m-1) = 0 \Rightarrow m_2 = 1$

$$y_1 = e^{0x} = 1 \quad \text{and} \quad y_2 = e^{1x} = e^x$$

$$y_c = C_1 + C_2 e^x$$

The general solution to the nonhomogeneous
equation is

$$y = C_1 + C_2 e^{-5x} + \frac{2}{15} e^{-5x} - 4 \sin(2x) + 2 \cos(2x).$$

A Glitch!

$$y'' - y' = 3e^x$$

$$g(x) = 3e^x, \text{ try } y_p = Ae^x$$
$$y_p' = Ae^x$$
$$y_p'' = Ae^x$$

$$y_p'' - y_p' = 3e^x$$

this is false for all choices of A

$$Ae^x - Ae^x = 3e^x$$
$$0e^x = 3e^x \quad \leftarrow$$

Recall $y_c = C_1 + C_2 e^x$

There is a fix: Multiply Ae^x by x^n where n is the smallest positive integer such that $Ax^n e^x$ does not solve the homogeneous equation.

Start w/ Ae^x multiply by x

$$y_p = Axe^x \quad \leftarrow \begin{matrix} \text{Does not} \\ \text{solve} \\ y'' - y' = 0 \\ \text{it will work.} \end{matrix}$$

$$y_p' = Ae^x + Axe^x, \quad y_p'' = Ae^x + Ae^x + Axe^x = 2Ae^x + Axe^x$$

$$y_p'' - y_p' = 3e^x$$

$$2Ae^x + Axe^x - (Ae^x + Axe^x) = 3e^x$$

$$(A - A)x e^x + (2A - A)e^x = 3e^x$$

$$Ae^x = 3e^x \Rightarrow A = 3$$

$$\text{so } y_p = 3xe^x$$

The general solution is

$$y = C_1 + C_2 e^x + 3xe^x$$

We'll consider cases

Using superposition as needed, begin with assumption:

$$y_p = y_{p_1} + \cdots + y_{p_k}$$

where y_{p_i} has the same **general form** as $g_i(x)$.

Case I: y_p as first written has no part that duplicates the complementary solution y_c . Then this first form will suffice.

Case II: y_p has a term y_{p_i} that duplicates a term in the complementary solution y_c . Multiply that term by x^n , where n is the smallest positive integer that eliminates the duplication.

Find the form of the particular solution

$$y'' - 4y' + 4y = \sin(4x) + xe^{2x}$$

Let $y_1 = \sin 4x$, $y_2 = xe^{2x}$

at first guess

$$y_{p_1} = A \sin 4x + B \cos 4x$$

$$y_{p_2} = (Cx+D)e^{2x}$$

Consider y_c : $y'' - 4y' + 4y = 0$

$$m^2 - 4m + 4 = 0 \Rightarrow (m-2)^2 = 0$$

$m=2$, repeated root

$$y_1 = e^{2x}, \quad y_2 = xe^{2x}$$

$$y_c = C_1 e^{2x} + C_2 x e^{2x}$$

y_{p_1} requires no modification

$$y_{p_2} = (Cx + D) \cdot e^{2x} = Cx e^{2x} + D e^{2x}$$

wont work

$$\text{try } y_{p_2} = (Cx + D)x e^{2x} = Cx^2 e^{2x} + Dx e^{2x}$$

still wont work

$$y_{p_2} = (Cx + D)x^2 e^{2x} = Cx^3 e^{2x} + Dx^2 e^{2x}$$

this will work!

So together

$$y_p = A \sin(4x) + B \cos(4x) + (Cx^3 + Dx^2)e^{2x}$$