September 4 Math 2306 sec 51 Fall 2015

Section 3.1 (1.3, and a peek at 3.2) Applications

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

We converted the statement into the IVP

$$\frac{dA}{dt} + \frac{1}{100}A = 10, \quad A(0) = 0.$$

It's in standard form. $P(t) = \frac{1}{100}$ $\int P(t)\,dt = \frac{1}{100}t$

Integrating factor $\mu = e^{\int P(t)\,dt} = e^{\frac{1}{100}t}$

$$e^{\frac{1}{100}t}\frac{dA}{dt} + \frac{1}{100}e^{\frac{1}{100}t}A = 10e^{\frac{1}{100}t}$$

$$\frac{d}{dt}\left[e^{\frac{1}{100}t}A\right] = 10e^{\frac{1}{100}t}$$

$$\int \frac{d}{dt}\left[e^{\frac{1}{100}t}A\right]dt = \int 10e^{\frac{1}{100}t}\,dt$$

$$e^{\frac{1}{100}t}A = 10\cdot 100\,e^{\frac{1}{100}t} + C$$

$$A = 1000 + Ce^{-\frac{1}{100}t}$$

$A(0)=0 \quad \Rightarrow \quad A(0) = 1000 + Ce^{0} = 0$

$\Rightarrow \quad C = -1000$

$$\boxed{\text{The amount of salt } A(t) = 1000 - 1000\, e^{\frac{-1}{100}t}.}$$

At $t=5$, $A(t) = 1000 - 1000\, e^{\frac{-1}{100}\cdot 5}$

$= 1000\left(1 - e^{\frac{-1}{20}}\right) \approx 48.8$ lbs

The Volume is 500 gal

So at $t=5$ the concentration

$$C = \frac{A(5)}{V} = \frac{48.8 \text{ lb}}{500 \text{ gal}} \approx 0.098 \frac{\text{lb}}{\text{gal}}$$

$r_i \neq r_o$

Suppose that instead, the mixture is pumped out at 10 gal/min. Determine the differential equation satisfied by $A(t)$ under this new condition.

<u>Incoming</u>:

rate $r_i = 5 \frac{\text{gal}}{\text{min}}$

Conc. $c_i = 2 \frac{\text{lb}}{\text{gal}}$

Outgoing

rate: $r_o = 10 \frac{\text{gal}}{\text{min}}$

Conc.: $c_o = \frac{A(t)}{V}$

Volume in the tank

Volume @ time t $\quad V(t) = V(0) + r_i\, t - r_o\, t$

$$= 500 \text{ gal} + 5 \frac{\text{gal}}{\text{min}} t \text{ min} - 10 \frac{\text{gal}}{\text{min}} t \text{ min}$$

$$= 500 - 5t$$

So $\quad C_o = \dfrac{A}{500-5t}$

$$\frac{dA}{dt} = r_i C_i - r_o C_o$$

$$\frac{dA}{dt} = 2 \frac{\text{lb}}{\text{gal}} \cdot 5 \frac{\text{gal}}{\text{min}} - 10 \frac{\text{gal}}{\text{min}} \cdot \frac{A \text{ lb}}{500-5t \text{ gal}}$$

$$\frac{dA}{dt} = 10 - \frac{2A}{100-t} \implies \boxed{\frac{dA}{dt} + \frac{2}{100-t} A = 10}$$

A Nonlinear Modeling Problem

A population $P(t)$ of tilapia changes at a rate jointly proportional to the current population and the difference between the constant carrying capacity[1] M of the environment and the current population. Determine the differential equation satsified by P.

rate of change of P is $\frac{dP}{dt}$

jointly proportional to P and the difference of M and P

$$kP(M-P)$$

$$\frac{dP}{dt} = kP(M-P)$$

[1] The carrying capacity is the maximum number of individuals that the environment can support due to limitation of space and resources.

Logistic Differential Equation

The equation

$$\frac{dP}{dt} = kP(M-P), \quad k, M > 0$$

is called a **logistic growth equation.**

Solve this equation[2] and show that for any $P(0) > 0$, $P \to M$ as $t \to \infty$.

$$\frac{dP}{dt} = kP(M-P) \quad , \quad P(0) = P_0 \quad \text{for some } P_0 > 0.$$

Separable

$$\frac{1}{P(M-P)} \frac{dP}{dt} = k \, dt$$

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[2] The partial fraction decomposition

$$\frac{1}{P(M-P)} = \frac{1}{M}\left(\frac{1}{P} + \frac{1}{M-P}\right)$$

is useful.

$$\int \frac{1}{P(M-P)}\, dP = \int k\, dt$$

$$\frac{1}{M} \int \left(\frac{1}{P} + \frac{1}{M-P} \right) dP = \int k\, dt$$

$$\int \left(\frac{1}{P} + \frac{1}{M-P} \right) dP = \int Mk\, dt = Mkt + C$$

$$\ln|P| + \ln|M-P| = Mkt + C$$

We'll complete this problem next time.