September 4 Math 2306 sec 54 Fall 2015

Section 3.1 (1.3, and a peek at 3.2) Applications

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

We converted the statement into the IVP

$$\frac{dA}{dt} + \frac{1}{100}A = 10, \quad A(0) = 0.$$

It's in standard form. $P(t) = \frac{1}{100}$

$\int P(t)\,dt = \int \frac{1}{100}\,dt = \frac{1}{100}t$, $\mu = e^{\int P(t)\,dt} = e^{\frac{1}{100}t}$

$$e^{\frac{1}{100}t}\frac{dA}{dt} + \frac{1}{100}e^{\frac{1}{100}t}A = 10e^{\frac{1}{100}t}$$

$$\frac{d}{dt}\left[e^{\frac{1}{100}t}A\right] = 10e^{\frac{1}{100}t}$$

$$\int \frac{d}{dt}\left[e^{\frac{1}{100}t}A\right]dt = \int 10e^{\frac{1}{100}t}\,dt$$

$$e^{\frac{1}{100}t}A = 10\cdot 100\,e^{\frac{1}{100}t} + C$$

$$A = 1000 + Ce^{\frac{-1}{100}t}$$

$$A(0) = 0 \Rightarrow A(0) = 1000 + Ce^{0} = 0$$

$$C = -1000$$

$$\boxed{\text{The amount of salt in the tank at time } t \text{ is } A(t) = 1000 - 1000\, e^{-\frac{1}{100}t}.}$$

At $t=5$, $A(5) = 1000\left(1 - e^{-\frac{1}{100}\cdot 5}\right) = 1000\left(1 - e^{-0.05}\right)$

$$\approx 48.8$$

$V = 500$ gal

So the concentration in the tank @ $t = 5$ min

is

$$C = \frac{48.8 \text{ lb}}{500 \text{ gal}} \approx 0.0975 \frac{\text{lb}}{\text{gal}}$$

$r_i \neq r_o$

Suppose that instead, the mixture is pumped out at 10 gal/min. Determine the differential equation satisfied by $A(t)$ under this new condition.

Incoming:

rate : $r_i = 5 \, \frac{\text{gal}}{\text{min}}$

conc : $c_i = \frac{2\text{ lb}}{\text{gal}}$

Outgoing:

rate: $r_o = 10 \, \frac{\text{gal}}{\text{min}}$

Conc: $c_o = \frac{A(t)\text{ lb}}{V(t)\text{ gal}}$

$$V(t) = V(0)\text{ gal} + r_i \frac{\text{gal}}{\text{min}} t \text{ min} - r_o \frac{\text{gal}}{\text{min}} \cdot t \text{ min}$$

$$V(t) = 500 + 5t - 10t = 500 - 5t$$

$$\Rightarrow \quad c_0 = \frac{A}{500 - 5t}$$

$$\frac{dA}{dt} = r_i c_i - r_0 c_0 = 5 \cdot 2 - 10 \cdot \frac{A}{500 - 5t}$$

$$\frac{dA}{dt} + \frac{2}{100 - t} A = 10$$

A Nonlinear Modeling Problem

A population $P(t)$ of tilapia changes at a rate jointly proportional to the current population and the difference between the constant carrying capacity[1] M of the environment and the current population. Determine the differential equation satsified by P.

rate of change of P is $\frac{dP}{dt}$

jointly proportional to P and the difference between P and M is $kP(M-P)$

The DE is $\frac{dP}{dt} = kP(M-P)$.

[1] The carrying capacity is the maximum number of individuals that the environment can support due to limitation of space and resources.

Logistic Differential Equation

The equation

$$\frac{dP}{dt} = kP(M-P), \quad k, M > 0$$

is called a **logistic growth equation.**

Solve this equation[2] and show that for any $P(0) > 0$, $P \to M$ as $t \to \infty$.

$$\frac{dP}{dt} = kP(M-P) \Rightarrow \frac{1}{P(M-P)} \frac{dP}{dt}\, dt = k\, dt$$

$$\int \frac{1}{P(M-P)}\, dP = \int k\, dt$$

[2]The partial fraction decomposition

$$\frac{1}{P(M-P)} = \frac{1}{M}\left(\frac{1}{P} + \frac{1}{M-P}\right)$$

is useful.

$$\int \frac{1}{M}\left(\frac{1}{P} + \frac{1}{M-P}\right) dP = \int k\,dt$$

$$\int \left(\frac{1}{P} + \frac{1}{M-P}\right) dP = \int kM\,dt = kMt + C$$

$$\ln|P| - \ln|M-P| = kMt + C$$

We'll finish next time.