# September 5 MATH 1113 sec. 51 Fall 2018

## Chapters 3 & 4: Introduction to Algebraic Functions

For much of the remainder of this course, we will be studying functions that are commonly used in almost all disciplines to model processes. Functions that fall into the category of algebraic functions include, polynomials, rational functions, and functions that involve fractional powers. We'll start with

Polynomials.

# Polynomials

**Definition:** A **polynomial** in a variable $x$ is an expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,$$

where $n$ is a nonnegative integer, and the numbers $a_0, a_1, \ldots, a_n$ are called the *coefficients*.

We will assume that the coefficients are real numbers (typically known), and that $x$ is a real valued variable.

**Definition:** A **polynomial equation** is one of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = 0.$$

**Definition:** A **polynomial function** $f$ is one defined by

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0.$$

# Some Key Characteristics

Suppose $n \geq 1$, and $a_n \neq 0$. Given a polynomial function

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,$$

- the **degree** of $f$ is the value $n$.
- the number $a_n$ is called the **leading coefficient**, and the term $a_n x^n$ is called the **leading term**.
- the domain of a polynomial function is the set of all real numbers.
- Polynomial functions are *continuous* and *smooth*. The graph of a polynomial will not have any gaps/jumps, holes, vertical asymptotes, or sharp corners.

# Constant Functions

**Constant Functions:** If $a_0 \neq 0$, the constant function $f(x) = a_0$ is a polynomial of degree zero.

**The Zero Polynomial:** The function $f(x) = 0$ is a polynomial. It does not have a degree[1].

[1] Some author's will call the degree of the zero polynomial $-\infty$; other's will just say it is undefined.

# Question

Recall that a polynomial function has the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0.$$

$g(x) = a_1 x^1 + a_0$ where $a_1 = 2$ and $a_0 = -1$

Let $g(x) = 2x - 1$. Which of the following is true?

(a) The function $g$ IS NOT a polynomial because it is not called $f(x)$.

(b) The function $g$ IS a polynomial function, and its degree is 1.

(c) The function $g$ IS NOT a polynomial because its graph is a line.

(d) The function $g$ IS a polynomial function, but it does not have a degree.

(e) The function $g$ IS NOT a polynomial because it doesn't have an $x^2$ term.

# Lines and Quadratics

A linear function $f(x) = mx + b$ is a polynomial. If $m \neq 0$, then its degree is 1.

**Definition:** A quadratic function is a second degree polynomial $f(x) = ax^2 + bx + c$ where $a \neq 0$.

It is assumed that we are already familiar with the basics of quadratic functions and quadratic equations. Here, we will review the main properties.

# Section 3.2 & 3.3: Quadratic Functions and Quadratic Equations

**Definition:** A quadratic equation in $x$ is one that can be written in the form

$$ax^2 + bx + c = 0, \quad \text{where} \quad a \neq 0. \tag{1}$$

We will say that the above is written in **standard form**.

If (1) has a real number solution $x_0$, then this number is called

- a **root** of the equation (1).
- It is also called a **zero** of the associated quadratic function $f(x) = ax^2 + bx + c$.
- A zero of $f(x) = ax^2 + bx + c$ is the location of an $x$-**intercept** to the graph of $f$.

# Parabola

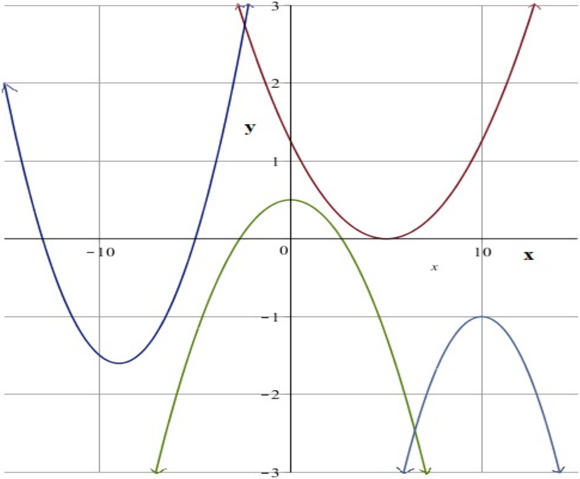


Figure: The graph of a quadratic is a parabola. It may open upward $(a > 0)$ or downward $(a < 0)$. It may have zero, one, or two $x$-intercepts.

# Parabola

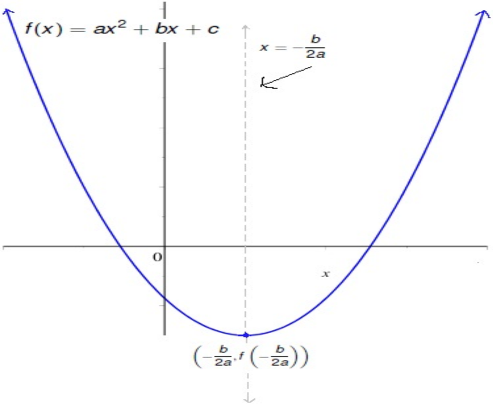


Figure: The graph of a quadratic function has a relative maximum or minimum called the **vertex** and is symmetric about the line $x = -\dfrac{b}{2a}$. Every quadratic function can be graphed using transformations on the graph of $y = x^2$.

# Finding Roots of Quadratics

First, a couple of theorems:

**Theorem: The Zero Product Property** For real numbers $a$ and $b$,

$$ab = 0 \quad \text{if and only if} \quad a = 0 \quad \text{or} \quad b = 0.$$

**Theorem: Principle of Square Roots** For nonnegative real number $k$,

$$x^2 = k \quad \text{if and only if} \quad x = \sqrt{k} \quad \text{or} \quad x = -\sqrt{k}.$$

# Example

Find the $x$-intercepts of the fuction $f(x) = 2x^2 + 3x - 2$.

It helps to note that $f(x) = (2x-1)(x+2)$

We want to find $x$-value(s) such that $f(x) = 0$.

If $f(x) = 0$, then

$(2x-1)(x+2) = 0$ By the zero product

property $2x - 1 = 0$ or $x + 2 = 0$.

$x = \frac{1}{2}$ or $x = -2$

The $x$-intercepts are $\left(\frac{1}{2}, 0\right)$ and $(-2, 0)$.

# Question

Suppose we wish to solve the quadratic equation $x^2 + x = 6$, and we factor the left side as

$$x(x+1) = 6.$$

**True/False** It follows that $x = 6$ or $x + 1 = 6$.

(a) True, but I'm not confident.

(b) True, and I am confident.

(c) False, but I'm not confident.

(d) False, and I am confident.

This is false.

This is a confusion of the zero product property.

To use the 0-product property, write

$$x^2 + x - 6 = 0$$

then try to factor.

# Completing the Square

To solve quadratic equations or to plot the graph of a quadratic function, it is useful to express a quadratic in a new form sometimes called **vertex form.** We wish to write $ax^2 + bx + c$ as

$$ax^2 + bx + c = a(x-h)^2 + k$$

If $f(x) = a(x-h)^2 + k$, then the graph of $f$ has vertex at the point $(h, k)$ and is symmetric about the vertical line $x = h$. Moreover, if $f$ has a real zero $x_0$, then

$$x_0 = h + \sqrt{-\frac{k}{a}} \quad \text{or} \quad x_0 = h - \sqrt{-\frac{k}{a}}.$$

# Completing the Square

If $f(x) = a(x-h)^2 + k$ has a real zero $x_0$, show that    here $a \neq 0$

$$x_0 = h + \sqrt{-\frac{k}{a}} \quad \text{or} \quad x_0 = h - \sqrt{-\frac{k}{a}}.$$

If $f(x_0) = 0$ then $a(x_0 - h)^2 + k = 0$

$(x_0 - h)^2 = \frac{-k}{a}$    using the priciple of square roots

$x_0 - h = \sqrt{\frac{-k}{a}}$ or $x_0 - h = -\sqrt{\frac{-k}{a}}$

i.e.

$x_0 = h + \sqrt{\frac{-k}{a}}$ or $x_0 = h - \sqrt{\frac{-k}{a}}$

# Completing the Square[2]

Assuming $a \neq 0$, determine $h$ and $k$ in terms of $a$, $b$, and $c$ such that $ax^2 + bx + c = a(x-h)^2 + k$.

$$ax^2+bx+c = a(x^2-2hx+h^2)+k$$

$$= ax^2 - 2hax + ah^2 + k$$

Match coefficients:

$x^2$ $\quad a = a \quad$ done

$x$ $\quad b = -2ha \quad \Rightarrow \quad h = \frac{-b}{2a}$

Constant $\quad c = ah^2 + k$

[2] Theorem: Two polynomials are equal if and only if they have equal corresponding coefficients.

So

$$k = c - ah^2 = c - a\left(\frac{-b}{2a}\right)^2$$

$$= c - a\frac{b^2}{4a^2} = c - \frac{b^2}{4a}$$

$$= \frac{4ca - b^2}{4a}$$

So

$$h = \frac{-b}{2a} \quad \text{and} \quad k = \frac{4ca - b^2}{4a}$$