August 31 Math 2306 sec. 53 Fall 2018

Section 5: First Order Equations Models and Applications

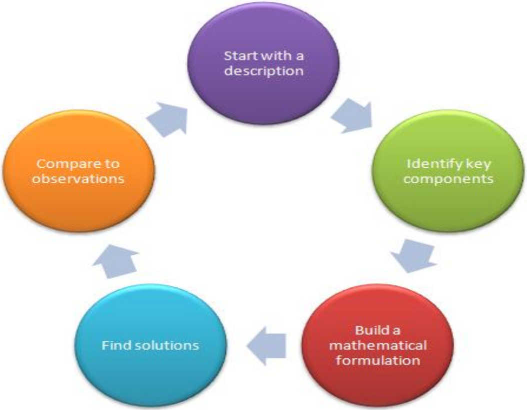

Figure: Mathematical Models give Rise to Differential Equations

Population Dynamics

A population of dwarf rabbits grows at a rate proportional to the current population. In 2011, there were 58 rabbits. In 2012, the population was up to 89 rabbits. Estimate the number of rabbits expected in the population in 2021.

Let P be the population density (no. rabbits per unit habitat) at time t. So $P = P(t)$. Let t be in years since 2011 so $t=0$ in 2011.

We're told the rate of change of P is proportional to P.

$$\frac{dP}{dt} \propto P$$

rate of change of P

P

So $\frac{dP}{dt} = kP$ for some constant k.

This is a separable ODE for P (it's also linear).

From the statement,

$$P(0) = 58 \quad \text{and} \quad P(1) = 89$$

We have an IVP

$$\frac{dP}{dt} = kP, \quad P(0) = 58$$

Let's solve this for $P(t)$.

Separate variables

$$\frac{1}{P}\frac{dP}{dt} = k \implies \int \frac{1}{P}\,dP = \int k\,dt$$

$$\ln P = kt + C \implies P = e^{kt+C} = e^{kt}\cdot e^{C}$$

Letting $A = e^{C}$, we have

$$P(t) = Ae^{kt}$$

from $P(0)=58$, $\quad 58 = Ae^{0} = A \implies A = 58$

so $\quad P(t) = 58e^{kt}$

From $P(1) = 89 \quad 89 = 58e^{k\cdot 1} = 58e^{k}$

So $e^k = \frac{89}{58} \implies k = \ln\left(\frac{89}{58}\right)$

The population is therefore

$$P(t) = 58\, e^{t \ln\left(\frac{89}{58}\right)}$$

In 2021 (i.e. $t = 10$)

$$P(10) = 58\, e^{10 \ln\left(\frac{89}{58}\right)} \approx 4198.05$$

This predicts just under 4200 rabbits in 2021.

Exponential Growth or Decay

If a quantity P changes continuously at a rate proportional to its current level, then it will be governed by a differential equation of the form

$$\frac{dP}{dt} = kP \quad \text{i.e.} \quad \frac{dP}{dt} - kP = 0.$$

Note that this equation is both separable and first order linear. If $k > 0$, P experiences **exponential growth**. If $k < 0$, then P experiences **exponential decay**.

Series Circuits: RC-circuit

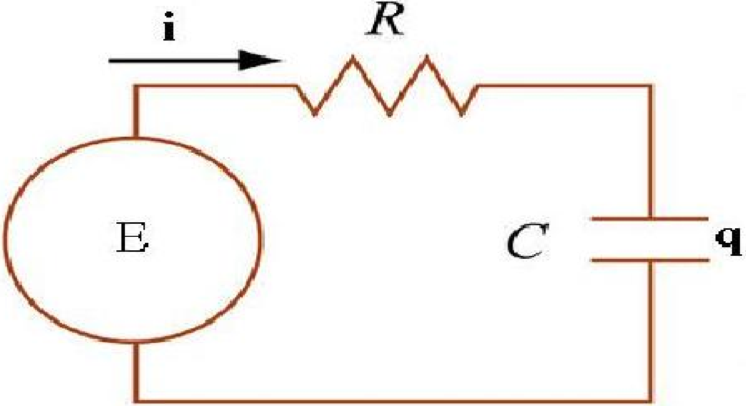

Figure: Series Circuit with Applied Electromotive force E, Resistance R, and Capcitance C. The charge of the capacitor is q and the current $i = \frac{dq}{dt}$.

Series Circuits: LR-circuit

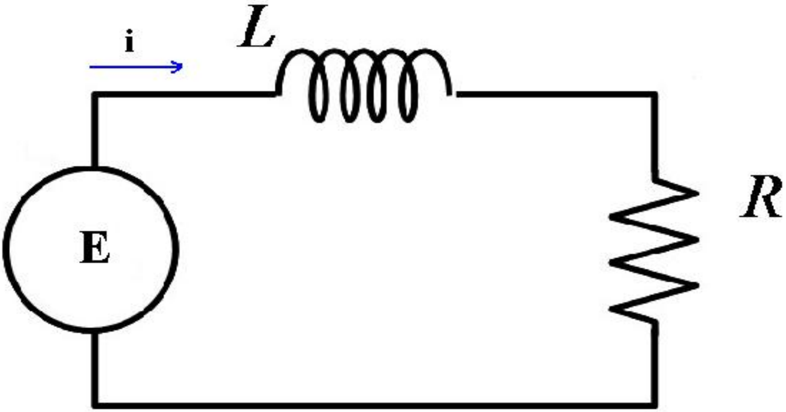

Figure: Series Circuit with Applied Electromotive force E, Inductance L, and Resistance R. The current is i.

Measurable Quantities:

Resistance R in ohms (Ω),	Implied voltage E in volts (V),
Inductance L in henries (h),	Charge q in coulombs (C),
Capacitance C in farads (f),	Current i in amperes (A)

Current is the rate of change of charge with respect to time: $i = \dfrac{dq}{dt}$.

Component	Potential Drop
Inductor	$L\frac{di}{dt}$
Resistor	Ri i.e. $R\frac{dq}{dt}$
Capacitor	$\frac{1}{C}q$

Kirchhoff's Law

The sum of the voltages around a closed circuit is zero.

In other words, the sum of potential drops across the passive components is equal to the applied electromotive force.

RC

PD resistor + PD capacitor = Applied force

$$R\frac{dq}{dt} + \frac{1}{C}q = E$$

1^{st} order linear eqn for $q(t)$.

LR

PD Inductor + PD resistor = Applied Force

$$L\frac{di}{dt} + Ri = E$$

1st order linear ODE for i