September 5 Math 2306 sec. 56 Fall 2017

Section 4: First Order Equations: Exact Equations

$$M\,dx + N\,dy = 0 \qquad (1)$$

Theorem: If $(\partial M/\partial y - \partial N/\partial x)/N$ is continuous and depends only on x, then

$$\mu = \exp\left(\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\,dx\right)$$

is an special integrating factor for (1). If $(\partial N/\partial x - \partial M/\partial y)/M$ is

continuous and depends only on y, then

$$\mu = \exp\left(\int \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}\,dy\right)$$

is an special integrating factor for (1).

Example

Solve the equation $2xy\,dx + (y^2 - 3x^2)\,dy = 0$.

We found that the equation was not exact and looked at the ratios:

Is there a $\mu(x)$?
$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{8x}{y^2 - 3x^2}$$

Is there a $\mu(y)$?
$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-8x}{2xy} = \frac{-4}{y}$$

The first does not depend on x alone. But the second does depend on y alone. So there is a special integrating factor

$$\mu = \exp\left(\int -\frac{4}{y}\,dy\right) = y^{-4}$$

Mult. by integrating factor

$$y^{-4}\left(2xy\,dx + (y^2-3x^2)\,dy\right) = 0\cdot y^{-4}$$

$$2xy^{-3}\,dx + (y^{-2}-3x^2y^{-4})\,dy = 0$$

$$\frac{\partial(\mu M)}{\partial y} = -6xy^{-4} \qquad \frac{\partial(\mu N)}{\partial x} = -6xy^{-4}$$

Exact!

Find $F(x,y)$ with the property

$$\frac{\partial F}{\partial x} = 2xy^{-3} \quad \text{and} \quad \frac{\partial F}{\partial y} = y^{-2}-3x^2y^{-4}$$

$$F(x,y) = \int \frac{\partial F}{\partial x}\,dx = \int 2x\,y^{-3}\,dx$$

$$= 2\,\frac{x^2}{2}\,y^{-3} + g(y).$$

$$= x^2 y^{-3} + g(y)$$

$$\frac{\partial F}{\partial y} = -3x^2 y^{-4} + g'(y) = y^{-2} - 3x^2 y^{-4}$$

Matching $\Rightarrow$ $g'(y) = y^{-2}$

an antiderivative is $g(y) = \frac{y^{-1}}{-1} = \frac{-1}{y}$

So $F(x,y) = x^2y^{-3} - \frac{1}{y} = \frac{x^2}{y^3} - \frac{1}{y}$

Solutions to the ODE are given implicitly by

$$\frac{x^2}{y^3} - \frac{1}{y} = C.$$

Section 5: First Order Equations Models and Applications

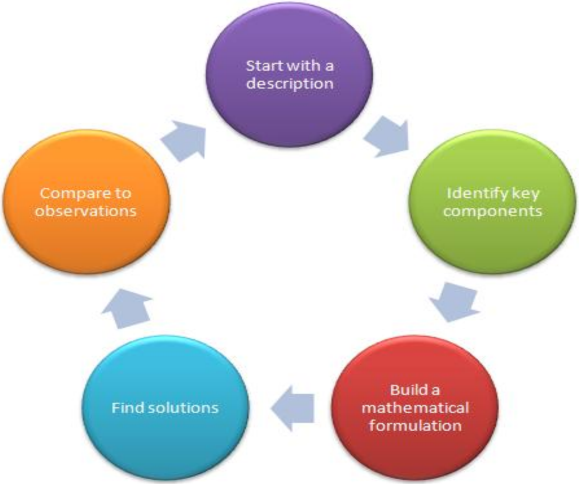

Figure: Mathematical Models give Rise to Differential Equations

Population Dynamics

A population of dwarf rabbits grows at a rate proportional to the current population. In 2011, there were 58 rabbits. In 2012, the population was up to 89 rabbits. Estimate the number of rabbits expected in the population in 2021.

Let $P(t)$ be the rabbit population (density) at time t.

The rate of change of P $\frac{dP}{dt}$ is proportional to P.

$$\frac{dP}{dt} = kP \quad \text{for some constant } k.$$

Let's take t in years w/ $t=0$ in 2011.

We know that $P(0)=58$ and $P(1)=89$

Let's solve the IVP $\frac{dP}{dt} = kP$, $P(0) = 58$.

Separate variables

$$\frac{1}{P}\frac{dP}{dt} = k \implies \int \frac{1}{P}\,dP = \int k\,dt$$

$$\ln P = kt + C$$

* Note: we're assuming $P > 0$.

$$e^{\ln P} = e^{kt+C} = Ae^{kt} \quad \text{where } A = e^{C}$$

$$P = Ae^{kt}$$

$P(0) = 58 \quad \Rightarrow \quad 58 = Ae^{k\cdot 0} = A$

So $\quad P(t) = 58\, e^{kt}$

Using $P(1) = 89$

$$P(1) = 89 = 58\, e^{k\cdot 1} = 58\, e^{k}$$

$$e^{k} = \frac{89}{58} \quad \Rightarrow \quad k = \ln\left(\frac{89}{58}\right)$$

So

$$P(t) = 58\, e^{\ln\left(\frac{89}{58}\right)t} \qquad \text{for all } t \geq 0.$$

In 2021, $t=10$,

$$P(10) = 58\, e^{\ln\left(\frac{89}{58}\right)\cdot 10} \approx 4198$$

Exponential Growth or Decay

If a quantity P changes continuously at a rate proportional to its current level, then it will be governed by a differential equation of the form

$$\frac{dP}{dt} = kP \quad \text{i.e.} \quad \frac{dP}{dt} - kP = 0.$$

Note that this equation is both separable and first order linear. If $k > 0$, P experiences **exponential growth**. If $k < 0$, then P experiences **exponential decay**. The solution is always

$$P(t) = P(0)\, e^{kt}$$

Series Circuits: RC-circuit

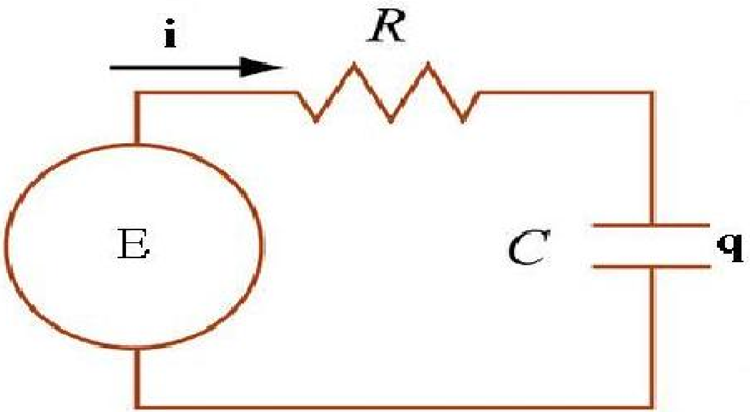

Figure: Series Circuit with Applied Electromotive force E, Resistance R, and Capcitance C. The charge of the capacitor is q and the current $i = \frac{dq}{dt}$.

Series Circuits: LR-circuit

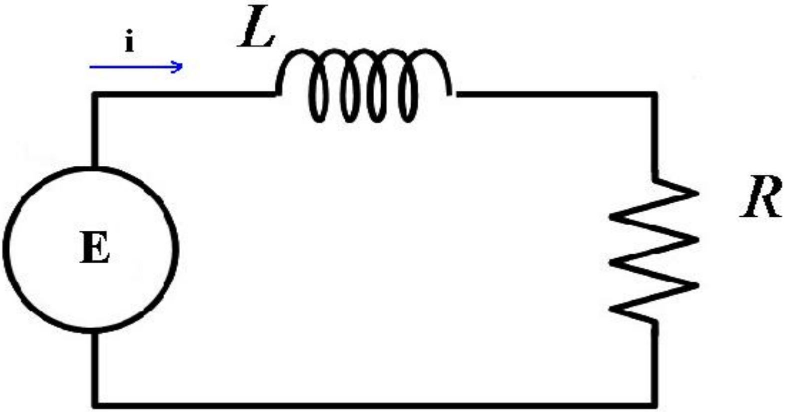

Figure: Series Circuit with Applied Electromotive force E, Inductance L, and Resistance R. The current is i.

Measurable Quantities:

Resistance R in ohms (Ω), Implied voltage E in volts (V),
Inductance L in henries (h), Charge q in coulombs (C),
Capacitance C in farads (f), Current i in amperes (A)

Current is the rate of change of charge with respect to time: $i = \dfrac{dq}{dt}$.

Component	Potential Drop
Inductor	$L\frac{di}{dt}$
Resistor	Ri i.e. $R\frac{dq}{dt}$
Capacitor	$\frac{1}{C}q$

Kirchhoff's Law

The sum of the voltages around a closed circuit is zero.

In other words, the sum of potential drops across the passive components is equal to the applied electromotive force.

RC circuit:

resistor capacitor

$$R\frac{dq}{dt} + \frac{1}{C}q = E$$

Constant coef. left side. 1st order linear.

LR circuit:

Inductor Resistor

$$L\frac{di}{dt} + Ri = E$$

1st order linear.

Example

A 200 volt battery is applied to an RC series circuit with resistance 1000Ω and capacitance 5×10^{-6} f. Find the charge $q(t)$ on the capacitor if $i(0) = 0.4A$. Determine the charge as $t \to \infty$.

$$R\frac{dq}{dt} + \frac{1}{C}q = E$$

$R = 1000\,\Omega \qquad C = 5\cdot 10^{-6}\ f$

$E = 200\ V \qquad i(0) = 0.4\ A = q'(0).$

$$1000\frac{dq}{dt} + \frac{1}{5\cdot 10^{-6}}q = 200$$

$$\frac{dq}{dt} + \frac{10^6}{5\cdot 1000}q = \frac{200}{1000}$$

$$\frac{10^6}{5\cdot 1000} = \frac{10^3}{5} = 200$$

$$\frac{dq}{dt} + 200q = \frac{1}{5}, \qquad q'(0) = \frac{2}{5}$$

$$P(t) = 200 \Rightarrow \mu = e^{\int P(t)\,dt} = e^{\int 200\,dt} = e^{200t}$$

$$e^{200t}\left(q' + 200q\right) = \frac{1}{5}e^{200t}$$

$$\frac{d}{dt}\left(e^{200t}q\right) = \frac{1}{5}e^{200t}$$

$$\int \frac{d}{dt}\left(e^{200t}q\right)dt = \int \frac{1}{5}e^{200t}\,dt$$

$$e^{200t} q = \frac{1}{5\cdot 200} e^{200t} + k$$

$$q = \frac{\frac{1}{1000} e^{200t} + k}{e^{200t}}$$

$$q = \frac{1}{1000} + k e^{-200t}$$

$$q'(t) = -200 k e^{-200t}$$

$$q'(0) = \frac{2}{5} = -200 k \overbrace{e^{-200\cdot 0}}^{1}$$

$$\frac{2}{5} = -200k$$

$$k = \frac{2}{5(-200)} = -\frac{1}{500}$$

The charge on the capacitor is

$$q(t) = \frac{1}{1000} - \frac{1}{500}e^{-200t}.$$

The long time charge

$$\lim_{t\to\infty} q(t) = \lim_{t\to\infty}\left(\frac{1}{1000} - \frac{1}{500}e^{-200t}\right) = \frac{1}{1000} \text{ Coulombs.}$$

A Classic Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

A Classic Mixing Problem

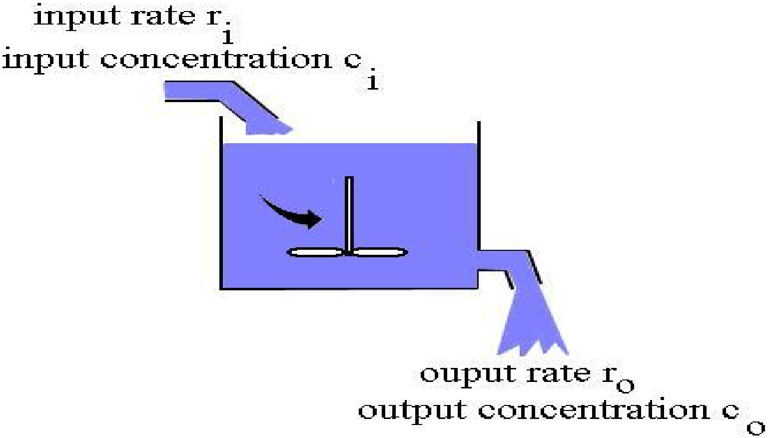

Figure: Spatially uniform composite fluids (e.g. salt & water, gas & ethanol) being mixed. Concentrations of substance change in time.

Building an Equation

The rate of change of the amount of salt

$$\frac{dA}{dt} = \left(\begin{array}{c} \text{input rate} \\ \text{of salt} \end{array}\right) - \left(\begin{array}{c} \text{output rate} \\ \text{of salt} \end{array}\right)$$

i - in
o - out

The input rate of salt is

$$\text{fluid rate in} \cdot \text{concentration of inflow} \quad = \quad r_i(c_i).$$

The output rate of salt is

$$\text{fluid rate out} \cdot \text{concentration of outflow} \quad = \quad r_o(c_o).$$

Building an Equation

The concentration of the outflowing fluid is

$$c_o = \frac{\text{total salt}}{\text{total volume}} = \frac{A(t)}{V(t)} = \frac{A(t)}{V(0) + (r_i - r_o)t}.$$

$$\frac{dA}{dt} = r_i \cdot c_i - r_o \frac{A}{V}.$$

This equation is first order linear.

$$\frac{dA}{dt} + \frac{r_o}{V} A = r_i \cdot c_i$$

Solve the Mixing Problem

A tank originally contains 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped in at a rate of 5gal/min. The well mixed solution is pumped out at the same rate. Find the amount of salt $A(t)$ in pounds at the time t. Find the concentration of the mixture in the tank at $t = 5$ minutes.

$r_i = 5$ gal/min $r_o = 5$ gal/min

$c_i = 2$ lb/gal $c_o = \frac{A}{V} = \frac{A \text{ lb}}{500 \text{ gal}} = \frac{1}{500} A \frac{\text{lb}}{\text{gal}}$.

$V(t) = V(0) + (r_i - r_o)t = 500 + (5-5)t$

$A(0) = 0$ pure water.

We'll solve the IVP next Tuesday.