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Section 2.1: Matrix Operations

We can denote an $m \times n$ matrix A in one of several convenient forms

$$A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}]$$

where $a_{ij}, \quad i = 1, \ldots m, \; j = 1, \ldots n$ is the entry in row i and column j. We call the entries a_{ii} the main diagonal of the matrix.

Some Arithmetic

Scalar Multiplication: For $m \times n$ matrix $A = [a_{ij}]$ and scalar c

$$cA = [ca_{ij}].$$

Matrix Addition: For $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$

$$A + B = [a_{ij} + b_{ij}].$$

The sum of two matrices is only defined if they are of the same size.

Matrix Equality: Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal (i.e. $A = B$) provided

$$a_{ij} = b_{ij} \quad \text{for every} \quad i = 1, \ldots, m \quad \text{and} \quad j = 1, \ldots, n.$$

Theorem: Properties

The $m \times n$ **zero matrix** has a zero in each entry. We'll denote this matrix as O (or $O_{m,n}$ if the size is not clear from the context).

Theorem: Let A, B, and C be matrices of the same size and r and s be scalars. Then

(i) $A + B = B + A$	(iv) $r(A + B) = rA + rB$
(ii) $(A + B) + C = A + (B + C)$	(v) $(r + s)A = rA + sA$
(iii) $A + O = A$	(vi) $r(sA) = (rs)A = (sr)A$

Matrix Multiplication

We know that for any $m \times n$ matrix A, the operation **"multiply vectors in $\mathbb{R}^n$ by A"** defines a linear transformation (from $\mathbb{R}^n$ to $\mathbb{R}^m$).

We wish to define matrix multiplication in such a way as to correspond to **function composition.** Thus if

$$S(\mathbf{x}) = B\mathbf{x}, \quad \text{and} \quad T(\mathbf{v}) = A\mathbf{v},$$

then

$$(T \circ S)(\mathbf{x}) = T(S(\mathbf{x})) = A(B\mathbf{x}) = (AB)\mathbf{x}.$$

Matrix Multiplication

$$S : \mathbb{R}^p \longrightarrow \mathbb{R}^n \quad \Longrightarrow \quad B \sim n \times p$$
$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m \quad \Longrightarrow \quad A \sim m \times n$$
$$T \circ S : \mathbb{R}^p \longrightarrow \mathbb{R}^m \quad \Longrightarrow \quad AB \sim m \times p$$

$$B\mathbf{x} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_p\mathbf{b}_p \Longrightarrow$$
$$A(B\mathbf{x}) = x_1 A\mathbf{b}_1 + x_2 A\mathbf{b}_2 + \cdots + x_p A\mathbf{b}_p \Longrightarrow$$

$$AB = [A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \cdots \quad A\mathbf{b}_p]$$

The j^{th} column of AB is A times the j^{th} column of B.

Example

Compute the product AB where

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 6 \end{bmatrix}$$

Row-Column Rule for Computing the Matrix Product

If $AB = C = [c_{ij}]$, then

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

(The ij^{th} entry of the product is the *dot product* of i^{th} row of A with the j^{th} column of B.)

For example: $\begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 6 \end{bmatrix} =$

Theorem: Properties-Matrix Product

Let A be an $m \times n$ matrix. Let r be a scalar and B and C be matrices for which the indicated sums and products are defined. Then

(i) $A(BC) = (AB)C$

(ii) $A(B + C) = AB + AC$

(iii) $(B + C)A = BA + CA$

(iv) $r(AB) = (rA)B = A(rB)$, and

(v) $I_m A = A = AI_n$

Caveats!

(1) Matrix multiplication **does not** commute! In general $AB \neq BA$

(2) The zero product property **does not** hold! That is, if $AB = O$, one **cannot** conclude that one of the matrices A or B is a zero matrix.

(3) There is no *cancelation law*. That is, $AB = CB$ **does not** imply that A and C are equal.

Compute AB and BA where $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$.

Compute the products AB, CB, and BB where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Matrix Powers

If A is square—meaning A is an $n \times n$ matrix for some $n \geq 2$, then the product AA is defined. For positive integer k, we'll define

$$A^k = AA^{k-1}.$$

We define $A^0 = I_n$.

Transpose

Definition: Let $A = [a_{ij}]$ be an $m \times n$ matrix. The **transpose** of A is the $n \times m$ matrix denoted and defined by

$$A^T = [a_{ji}].$$

For example, if

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}, \quad \text{then} \quad A^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}.$$

Example

$$A = \begin{bmatrix} 5 & 5 \\ -1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 1 & 4 \end{bmatrix}$$

Compute A^T, B^T, the transpose of the product $(AB)^T$, and the product $B^T A^T$.

$$A = \begin{bmatrix} 5 & 5 \\ -1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 1 & 4 \end{bmatrix}$$

$$A = \begin{bmatrix} 5 & 5 \\ -1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 1 & 4 \end{bmatrix}$$

Theorem: Properties-Matrix Transposition

Let A and B be matrices such that the appropriate sums and products are defined, and let r be a scalar. Then

(i) $(A^T)^T = A$

(ii) $(A + B)^T = A^T + B^T$

(iii) $(rA)^T = rA^T$

(iv) $(AB)^T = B^T A^T$

Section 2.2: Inverse of a Matrix

Consider the scalar equation $ax = b$. Provided $a \neq 0$, we can solve this explicity

$$x = a^{-1}b$$

where a^{-1} is the unique number such that $aa^{-1} = a^{-1}a = 1$.

If A is an $n \times n$ matrix, we seek an analog A^{-1} that satisfies the condition

$$A^{-1}A = AA^{-1} = I_n.$$

If such matrix A^{-1} exists, we'll say that A is **nonsingular** (a.k.a. *invertible*). Otherwise, we'll say that A is **singular**.

Theorem (2×2 case)

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then A is singular.

The quantity $ad - bc$ is called the **determinant** of A and may be denoted in several ways

$$\det(A) \quad = \quad |A| \quad = \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

Find the inverse if possible

(a) $A = \begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix}$

(b) $A = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$

Theorem

If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Example

Solve the system

$$\begin{array}{rcrcr} 3x_1 & + & 2x_2 & = & -1 \\ -x_1 & + & 5x_2 & = & 4 \end{array}$$

Theorem

(i) If A is invertible, then A^{-1} is also invertible and

$$\left(A^{-1}\right)^{-1} = A.$$

(ii) If A and B are invertible $n \times n$ matrices, then the product AB is also invertible[1] with

$$(AB)^{-1} = B^{-1}A^{-1}.$$

(iii) If A is invertible, then so is A^T. Moreover

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T.$$

[1] This can generalize to the product of k invertible matrices.

Elementary Matrices

Definition: An **elementary** matrix is a square matrix obtained from the identity by performing one elementary row operation.

Examples:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Action of Elementary Matrices

Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, and compute the following products

$$E_1A, \quad E_2A, \quad \text{and} \quad E_3A.$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Remarks

- Elementary row operations can be equated with matrix multiplication (multiply on the left by an elementary matrix),
- Each elementary matrix is invertible where the inverse *undoes* the row operation,
- Reduction to rref is a sequence of row operations, so it is a sequence of matrix multiplications

$$\mathrm{rref}(A) = E_k \cdots E_2 E_1 A.$$

Theorem

An $n \times n$ matrix A is invertible if and only if it is row equivalent to the identity matrix I_n. Moreover, if

$$\mathrm{rref}(A) = E_k \cdots E_2 E_1 A = I_n, \quad \text{then} \quad A = (E_k \cdots E_2 E_1)^{-1} I_n.$$

That is,

$$A^{-1} = \left[(E_k \cdots E_2 E_1)^{-1} \right]^{-1} = E_k \cdots E_2 E_1.$$

The sequence of operations that reduces A to I_n, transforms I_n into A^{-1}.

This last observation—operations that take A to I_n also take I_n to A^{-1}—gives us a method for computing an inverse!

Algorithm for finding A^{-1}

To find the inverse of a given matrix A:

- Form the $n \times 2n$ augmented matrix $[A \quad I]$.
- Perform whatever row operations are needed to get the first n columns (the A part) to rref.
- If rref(A) is I, then $[A \quad I]$ is row equivalent to $[I \quad A^{-1}]$, and the inverse A^{-1} will be the last n columns of the reduced matrix.
- If rref(A) is NOT I, then A is not invertible.

Remarks: We don't need to know ahead of time if A is invertible to use this algorithm.
If A is singular, we can stop as soon as it's clear that rref$(A) \neq I$.

Examples: Find the Inverse if Possible

(a) $\begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 3 \\ -2 & -6 & 4 \end{bmatrix}$

Examples: Find the Inverse if Possible

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$

Solve the linear system if possible

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & 3x_3 & = & 3 \\ & & x_2 & + & 4x_3 & = & 3 \\ 5x_1 & + & 6x_2 & & & = & 4 \end{array}$$