Section 2.1: Matrix Operations

We can denote an $m \times n$ matrix $A$ in one of several convenient forms

$$A = [a_1 \ a_2 \ \cdots \ a_n] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}]$$

where $a_{ij}, \quad i = 1, \ldots m, \ j = 1, \ldots n$ is the entry in row $i$ and column $j$. We call the entries $a_{ii}$ the main diagonal of the matrix.
Some Arithmetic

Scalar Multiplication: For $m \times n$ matrix $A = [a_{ij}]$ and scalar $c$

\[ cA = [ca_{ij}] \].

Matrix Addition: For $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$

\[ A + B = [a_{ij} + b_{ij}] \].

The sum of two matrices is only defined if they are of the same size.

Matrix Equality: Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal (i.e. $A = B$) provided

\[ a_{ij} = b_{ij} \quad \text{for every} \quad i = 1, \ldots, m \quad \text{and} \quad j = 1, \ldots, n. \]
Theorem: Properties

The $m \times n$ zero matrix has a zero in each entry. We’ll denote this matrix as $O$ (or $O_{m,n}$ if the size is not clear from the context).

**Theorem:** Let $A$, $B$, and $C$ be matrices of the same size and $r$ and $s$ be scalars. Then

(i) $A + B = B + A$
(ii) $(A + B) + C = A + (B + C)$
(iii) $A + O = A$
(iv) $r(A + B) = rA + rB$
(v) $(r + s)A = rA + sA$
(vi) $r(sA) = (rs)A = (sr)A$
Matrix Multiplication

We know that for any $m \times n$ matrix $A$, the operation "multiply vectors in $\mathbb{R}^n$ by $A$" defines a linear transformation (from $\mathbb{R}^n$ to $\mathbb{R}^m$).

We wish to define matrix multiplication in such a way as to correspond to function composition. Thus if

$$S(x) = Bx, \quad \text{and} \quad T(v) = Av,$$

then

$$(T \circ S)(x) = T(S(x)) = A(Bx) = (AB)x.$$
Matrix Multiplication

\[ S : \mathbb{R}^p \rightarrow \mathbb{R}^n \implies B \sim n \times p \]

\[ T : \mathbb{R}^n \rightarrow \mathbb{R}^m \implies A \sim m \times n \]

\[ T \circ S : \mathbb{R}^p \rightarrow \mathbb{R}^m \implies AB \sim m \times p \]

\[ Bx = x_1 b_1 + x_2 b_2 + \cdots + x_p b_p \implies \]

\[ A(Bx) = x_1 A b_1 + x_2 A b_2 + \cdots + x_p A b_p \implies \]

\[ AB = [A b_1 \quad A b_2 \quad \cdots \quad A b_p] \]

The \( j^{th} \) column of \( AB \) is \( A \) times the \( j^{th} \) column of \( B \).
Example

Compute the product $AB$ where

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 6 \end{bmatrix}$$
Row-Column Rule for Computing the Matrix Product

If $AB = C = [c_{ij}]$, then

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$ 

(The $ij^{th}$ entry of the product is the dot product of $i^{th}$ row of $A$ with the $j^{th}$ column of $B$.)

For example:

$$\begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 6 \end{bmatrix} =$$
Theorem: Properties-Matrix Product

Let $A$ be an $m \times n$ matrix. Let $r$ be a scalar and $B$ and $C$ be matrices for which the indicated sums and products are defined. Then

(i) $A(BC) = (AB)C$

(ii) $A(B + C) = AB + AC$

(iii) $(B + C)A = BA + CA$

(iv) $r(AB) = (rA)B = A(rB)$, and

(v) $I_mA = A = AI_n$
Caveats!

(1) Matrix multiplication **does not** commute! In general $AB \neq BA$

(2) The zero product property **does not** hold! That is, if $AB = O$, one **cannot** conclude that one of the matrices $A$ or $B$ is a zero matrix.

(3) There is no *cancelation law*. That is, $AB = CB$ **does not** imply that $A$ and $C$ are equal.
Compute $AB$ and $BA$ where $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$. 
Compute the products $AB$, $CB$, and $BB$ where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$,

$B = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. 
Matrix Powers

If $A$ is square—meaning $A$ is an $n \times n$ matrix for some $n \geq 2$, then the product $AA$ is defined. For positive integer $k$, we’ll define

$$A^k = AA^{k-1}.$$ 

We define $A^0 = I_n$. 
**Transpose**

**Definition:** Let $A = [a_{ij}]$ be an $m \times n$ matrix. The **transpose** of $A$ is the $n \times m$ matrix denoted and defined by

$$A^T = [a_{ji}].$$

For example, if

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix},$$

then

$$A^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}.$$
Example

\[ A = \begin{bmatrix} 5 & 5 \\ -1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 1 & 4 \end{bmatrix} \]

Compute \( A^T \), \( B^T \), the transpose of the product \( (AB)^T \), and the product \( B^T A^T \).
\[ A = \begin{bmatrix} 5 & 5 \\ -1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 1 & 4 \end{bmatrix} \]
\[ A = \begin{bmatrix}
5 & 5 \\
-1 & 4
\end{bmatrix}, \quad B = \begin{bmatrix}
2 & 0 & 3 \\
-1 & 1 & 4
\end{bmatrix} \]
Theorem: Properties-Matrix Transposition

Let $A$ and $B$ be matrices such that the appropriate sums and products are defined, and let $r$ be a scalar. Then

(i) $(A^T)^T = A$

(ii) $(A + B)^T = A^T + B^T$

(iii) $(rA)^T = rA^T$

(iv) $(AB)^T = B^T A^T$
Section 2.2: Inverse of a Matrix

Consider the scalar equation \( ax = b \). Provided \( a \neq 0 \), we can solve this explicitly

\[
x = a^{-1}b
\]

where \( a^{-1} \) is the unique number such that \( aa^{-1} = a^{-1}a = 1 \).

If \( A \) is an \( n \times n \) matrix, we seek an analog \( A^{-1} \) that satisfies the condition

\[
A^{-1}A = AA^{-1} = I_n.
\]

If such matrix \( A^{-1} \) exists, we’ll say that \( A \) is nonsingular (a.k.a. invertible). Otherwise, we’ll say that \( A \) is singular.
Theorem (2 × 2 case)

Let \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \). If \( ad - bc \neq 0 \), then \( A \) is invertible and

\[
A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.
\]

If \( ad - bc = 0 \), then \( A \) is singular.

The quantity \( ad - bc \) is called the **determinant** of \( A \) and may be denoted in several ways

\[
det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.
\]
Find the inverse if possible

(a) \[ A = \begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix} \]

(b) \[ A = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix} \]
Theorem

If $A$ is an invertible $n \times n$ matrix, then for each $b$ in $\mathbb{R}^n$, the equation $Ax = b$ has unique solution $x = A^{-1}b$. 
Example

Solve the system

\[3x_1 + 2x_2 = -1\]
\[-x_1 + 5x_2 = 4\]
Theorem

(i) If $A$ is invertible, then $A^{-1}$ is also invertible and

\[
\left( A^{-1} \right)^{-1} = A.
\]

(ii) If $A$ and $B$ are invertible $n \times n$ matrices, then the product $AB$ is also invertible\(^1\) with

\[
(AB)^{-1} = B^{-1} A^{-1}.
\]

(iii) If $A$ is invertible, then so is $A^T$. Moreover

\[
\left( A^T \right)^{-1} = \left( A^{-1} \right)^T.
\]

\(^1\)This can generalize to the product of $k$ invertible matrices.
Elementary Matrices

Definition: An elementary matrix is a square matrix obtained from the identity by performing one elementary row operation.

Examples:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$
Action of Elementary Matrices

Let \( A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \), and compute the following products

\[ E_1 A, \quad E_2 A, \quad \text{and} \quad E_3 A. \]
\[ E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \]
$$E_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
Remarks

- Elementary row operations can be equated with matrix multiplication (multiply on the left by an elementary matrix),

- Each elementary matrix is invertible where the inverse undoes the row operation,

- Reduction to rref is a sequence of row operations, so it is a sequence of matrix multiplications

\[
\text{rref}(A) = E_k \cdots E_2 E_1 A.
\]
Theorem

An $n \times n$ matrix $A$ is invertible if and only if it is row equivalent to the identity matrix $I_n$. Moreover, if

$$\text{rref}(A) = E_k \cdots E_2 E_1 A = I_n,$$
then

$$A = (E_k \cdots E_2 E_1)^{-1} I_n.$$

That is,

$$A^{-1} = \left[ (E_k \cdots E_2 E_1)^{-1} \right]^{-1} = E_k \cdots E_2 E_1.$$

The sequence of operations that reduces $A$ to $I_n$, transforms $I_n$ into $A^{-1}$.

This last observation—operations that take $A$ to $I_n$ also take $I_n$ to $A^{-1}$—gives us a method for computing an inverse!
Algorithm for finding $A^{-1}$

To find the inverse of a given matrix $A$:

- Form the $n \times 2n$ augmented matrix $[A \ I]$.
- Perform whatever row operations are needed to get the first $n$ columns (the $A$ part) to rref.
- If rref($A$) is $I$, then $[A \ I]$ is row equivalent to $[I \ A^{-1}]$, and the inverse $A^{-1}$ will be the last $n$ columns of the reduced matrix.
- If rref($A$) is NOT $I$, then $A$ is not invertible.

Remarks: We don’t need to know ahead of time if $A$ is invertible to use this algorithm. If $A$ is singular, we can stop as soon as it’s clear that rref($A$) $\neq I$. 
Examples: Find the Inverse if Possible

(a) $\begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 3 \\ -2 & -6 & 4 \end{bmatrix}$
Examples: Find the Inverse if Possible

(b) \[
\begin{bmatrix}
1 & 2 & 3 \\
0 & 1 & 4 \\
5 & 6 & 0 \\
\end{bmatrix}
\]
Solve the linear system if possible

\[
\begin{align*}
    x_1 + 2x_2 + 3x_3 &= 3 \\
    x_2 + 4x_3 &= 3 \\
    5x_1 + 6x_2 &= 4
\end{align*}
\]