#### September 5 Math 3260 sec. 58 Fall 2017

#### Section 2.1: Matrix Operations

We can denote an  $m \times n$  matrix A in one of several convenient forms

$$A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}]$$

where  $a_{ij}$ , i = 1, ..., m, j = 1, ..., n is the entry in row *i* and column *j*. We call the entries  $a_{ij}$  the main diagonal of the matrix.

#### Some Arithmetic Scalar Multiplication: For $m \times n$ matrix $A = [a_{ij}]$ and scalar c

 $cA = [ca_{ij}].$ 

**Matrix Addition:** For  $m \times n$  matrices  $A = [a_{ij}]$  and  $B = [b_{ij}]$ 

$$A+B=[a_{ij}+b_{ij}].$$

The sum of two matrices is only defined if they are of the same size.

**Matrix Equality:** Two matrices  $A = [a_{ij}]$  and  $B = [b_{ij}]$  are equal (i.e. A = B) provided

$$a_{ij} = b_{ij}$$
 for every  $i = 1, \dots, m$  and  $j = 1, \dots, n$ .

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#### **Theorem: Properties**

The  $m \times n$  **zero matrix** has a zero in each entry. We'll denote this matrix as O (or  $O_{m,n}$  if the size is not clear from the context).

**Theorem:** Let *A*, *B*, and *C* be matrices of the same size and *r* and *s* be scalars. Then

(i) 
$$A + B = B + A$$
  
(iv)  $r(A + B) = rA + rB$   
(ii)  $(A + B) + C = A + (B + C)$   
(v)  $(r + s)A = rA + sA$   
(iii)  $A + O = A$   
(vi)  $r(sA) = (rs)A = (sr)A$ 

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## Matrix Multiplication

We know that for any  $m \times n$  matrix A, the operation "**multiply vectors** in  $\mathbb{R}^n$  by A" defines a linear transformation (from  $\mathbb{R}^n$  to  $\mathbb{R}^m$ ).

We wish to define matrix multiplication in such a way as to correspond to **function composition**. Thus if

$$S(\mathbf{x}) = B\mathbf{x}$$
, and  $T(\mathbf{v}) = A\mathbf{v}$ ,

then

$$(T \circ S)(\mathbf{x}) = T(S(\mathbf{x})) = A(B\mathbf{x}) = (AB)\mathbf{x}.$$

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#### Matrix Multiplication

$$S: \mathbb{R}^{p} \longrightarrow \mathbb{R}^{n} \implies B \sim n \times p$$
$$T: \mathbb{R}^{n} \longrightarrow \mathbb{R}^{m} \implies A \sim m \times n$$
$$T \circ S: \mathbb{R}^{p} \longrightarrow \mathbb{R}^{m} \implies AB \sim m \times p$$

$$B\mathbf{x} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \dots + x_p\mathbf{b}_p \Longrightarrow$$
$$A(B\mathbf{x}) = x_1A\mathbf{b}_1 + x_2A\mathbf{b}_2 + \dots + x_pA\mathbf{b}_p \Longrightarrow$$

$$AB = [A\mathbf{b}_1 \ A\mathbf{b}_2 \ \cdots \ A\mathbf{b}_p]$$

The  $j^{th}$  column of *AB* is *A* times the  $j^{th}$  column of *B*.

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## Example

Compute the product AB where

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 6 \end{bmatrix}$$

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Row-Column Rule for Computing the Matrix Product If  $AB = C = [c_{ij}]$ , then

$$c_{ij}=\sum_{k=1}^n a_{ik}b_{kj}.$$

(The *ij*<sup>th</sup> entry of the product is the *dot* product of *i*<sup>th</sup> row of *A* with the  $j^{th}$  column of *B*.)

For example: 
$$\begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 2 \\ 1 & -4 & 6 \end{bmatrix} =$$

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### Theorem: Properties-Matrix Product

Let *A* be an  $m \times n$  matrix. Let *r* be a scalar and *B* and *C* be matrices for which the indicated sums and products are defined. Then

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(i) A(BC) = (AB)C

(ii) 
$$A(B+C) = AB + AC$$

(iii) 
$$(B+C)A = BA + CA$$

(iv) r(AB) = (rA)B = A(rB), and

(v)  $I_m A = A = A I_n$ 



(1) Matrix multiplication **does not** commute! In general  $AB \neq BA$ 

(2) The zero product property **does not** hold! That is, if AB = O, one **cannot** conclude that one of the matrices A or B is a zero matrix.

(3) There is no *cancelation law*. That is, AB = CB **does not** imply that *A* and *C* are equal.

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Compute *AB* and *BA* where 
$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$
 and  $B = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}$ .

Compute the products *AB*, *CB*, and *BB* where  $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ ,

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$$B = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}$$
, and  $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ .

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If *A* is square—meaning *A* is an  $n \times n$  matrix for some  $n \ge 2$ , then the product *AA* is defined. For positive integer *k*, we'll define

$$A^k = AA^{k-1}.$$

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We define  $A^0 = I_n$ .

#### Transpose

**Definition:** Let  $A = [a_{ij}]$  be an  $m \times n$  matrix. The **transpose** of A is the  $n \times m$  matrix denoted and defined by

$$A^T = [a_{ji}].$$

For example, if

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$
, then  $A^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$ .

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#### Example

$$A = \begin{bmatrix} 5 & 5 \\ -1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 1 & 4 \end{bmatrix}$$

Compute  $A^T$ ,  $B^T$ , the transpose of the product  $(AB)^T$ , and the product  $B^T A^T$ .

## $A = \begin{bmatrix} 5 & 5 \\ -1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 1 & 4 \end{bmatrix}$

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# $A = \begin{bmatrix} 5 & 5 \\ -1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 3 \\ -1 & 1 & 4 \end{bmatrix}$

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### Theorem: Properties-Matrix Transposition

Let A and B be matrices such that the appropriate sums and products are defined, and let r be a scalar. Then

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(i)  $(A^{T})^{T} = A$ 

(ii) 
$$(A+B)^{T} = A^{T} + B^{T}$$

(iii)  $(rA)^T = rA^T$ 

(iv)  $(AB)^T = B^T A^T$ 

#### Section 2.2: Inverse of a Matrix

Consider the scalar equation ax = b. Provided  $a \neq 0$ , we can solve this explicity

$$x = a^{-1}b$$

where  $a^{-1}$  is the unique number such that  $aa^{-1} = a^{-1}a = 1$ .

If A is an  $n \times n$  matrix, we seek an analog  $A^{-1}$  that satisfies the condition

$$A^{-1}A = AA^{-1} = I_n.$$

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If such matrix  $A^{-1}$  exists, we'll say that A is **nonsingular** (a.k.a. *invertible*). Otherwise, we'll say that A is **singular**.

## Theorem (2 × 2 case) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . If $ad - bc \neq 0$ , then A is invertible and $A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ .

If ad - bc = 0, then A is singular.

The quantity ad - bc is called the **determinant** of A and may be denoted in several ways

$$\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

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Find the inverse if possible

(a) 
$$A = \begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix}$$

(b) 
$$A = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$$

## Theorem

If *A* is an invertible  $n \times n$  matrix, then for each **b** in  $\mathbb{R}^n$ , the equation  $A\mathbf{x} = \mathbf{b}$  has unique solution  $\mathbf{x} = A^{-1}\mathbf{b}$ .

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#### Solve the system



#### Theorem

(i) If A is invertible, then  $A^{-1}$  is also invertible and

$$\left(A^{-1}\right)^{-1}=A.$$

(ii) If *A* and *B* are invertible  $n \times n$  matrices, then the product *AB* is also invertible<sup>1</sup> with

$$(AB)^{-1} = B^{-1}A^{-1}.$$

(iii) If A is invertible, then so is  $A^{T}$ . Moreover

$$\left(\boldsymbol{A}^{T}\right)^{-1} = \left(\boldsymbol{A}^{-1}\right)^{T}.$$

## **Elementary Matrices**

**Definition:** An **elementary** matrix is a square matrix obtained from the identity by performing one elementary row operation.

Examples:

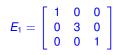
$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

.

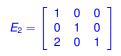
## Action of Elementary Matrices

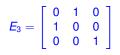
Let  $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ , and compute the following products

 $E_1A$ ,  $E_2A$ , and  $E_3A$ .



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- Elementary row operations can be equated with matrix multiplication (multiply on the left by an elementary matrix),
- Each elementary matrix is invertible where the inverse undoes the row operation,
- Reduction to rref is a sequence of row operations, so it is a sequence of matrix multiplications

$$\operatorname{rref}(A) = E_k \cdots E_2 E_1 A.$$

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#### Theorem

An  $n \times n$  matrix A is invertible if and only if it is row equivalent to the identity matrix  $I_n$ . Moreover, if

$$\operatorname{rref}(A) = E_k \cdots E_2 E_1 A = I_n$$
, then  $A = (E_k \cdots E_2 E_1)^{-1} I_n$ .

That is,

$$A^{-1} = \left[ (E_k \cdots E_2 E_1)^{-1} \right]^{-1} = E_k \cdots E_2 E_1.$$

The sequence of operations that reduces *A* to  $I_n$ , transforms  $I_n$  into  $A^{-1}$ .

This last observation—operations that take *A* to  $I_n$  also take  $I_n$  to  $A^{-1}$ —gives us a method for computing an inverse!

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## Algorithm for finding $A^{-1}$

To find the inverse of a given matrix A:

- Form the  $n \times 2n$  augmented matrix  $\begin{bmatrix} A & I \end{bmatrix}$ .
- Perform whatever row operations are needed to get the first n columns (the A part) to rref.
- If rref(A) is *I*, then [A I] is row equivalent to [I A<sup>-1</sup>], and the inverse A<sup>-1</sup> will be the last *n* columns of the reduced matrix.
- ▶ If rref(*A*) is NOT *I*, then *A* is not invertible.

**Remarks:** We don't need to know ahead of time if *A* is invertible to use this algorithm.

If A is singular, we can stop as soon as it's clear that  $rref(A) \neq I$ .

Examples: Find the Inverse if Possible

(a) 
$$\begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 3 \\ -2 & -6 & 4 \end{bmatrix}$$

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Examples: Find the Inverse if Possible

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(b) 
$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$$

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#### Solve the linear system if possible

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