Sept. 7 Math 1190 sec. 51 Fall 2016

Section 1.5: Infinite Limits, Limits at Infinity, Asymptotes

Definition: (Infinite Limits) Let $f(x)$ be defined on an open interval containing c except possibly at c. Then

$$\lim_{x\to c} f(x) = \infty$$

provided $f(x)$ can be made arbitrarily large by taking x sufficiently close to c. (The definition of

$$\lim_{x\to c} f(x) = -\infty$$

is similar except that f can be made arbitrarily large and negative.)

Limits at Infinity

Definitions: (Limits at Infinity) Let f be defined on an interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

provided the value of f can be made arbitrarily close to L by taking x sufficiently large.

Similarly

Defintion: Let f be defined on an interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

provided the value of f can be made arbitrarily close to L by taking x sufficiently large and negative.

Limits to Remember

Let k be any real number and let p be rational. Then

$$\lim_{x\to\infty} \frac{k}{x^p} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{k}{x^p} = 0$$

The latter holds assuming x^p is defined for $x < 0$.

In essence, if the numerator of a ratio is staying finite while the denominator is becoming infinite, the ratio is tending to zero.

Examples

(b) $\displaystyle\lim_{x\to-\infty}\frac{\sqrt{x^2+1}}{x+1}$

Mult. and divide by $\frac{1}{x}$

$= \displaystyle\lim_{x\to-\infty}\frac{\sqrt{x^2+1}}{x+1}\cdot\frac{\frac{1}{x}}{\frac{1}{x}}$

$= \displaystyle\lim_{x\to-\infty}\frac{\frac{1}{x}\sqrt{x^2+1}}{1+\frac{1}{x}}$

we're taking $x\to-\infty$

So we only really care about negative x

If $x<0$ then $|x|=-x$

$\sqrt{x^2}=|x|$

so $\frac{1}{\sqrt{x^2}}=\frac{1}{|x|}=\frac{1}{-x}$ for $x<0$

$$= \lim_{x \to -\infty} \frac{\frac{-1}{-x}\sqrt{x^2+1}}{1+\frac{1}{x}}$$

We need $\frac{1}{-x}$ to get $\frac{1}{x}$ with a square root.

$$= \lim_{x \to -\infty} - \frac{\frac{1}{-x}\sqrt{x^2+1}}{1+\frac{1}{x}}$$

$$= \lim_{x \to -\infty} - \frac{\frac{1}{\sqrt{x^2}}\sqrt{x^2+1}}{1+\frac{1}{x}}$$

$$= \lim_{x \to -\infty} - \frac{\sqrt{\frac{1}{x^2}(x^2+1)}}{1+\frac{1}{x}} = \lim_{x \to -\infty} - \frac{\sqrt{1+1/x^2}}{1+\frac{1}{x}} = -\frac{\sqrt{1+0}}{1+0} = -1$$

(c) $\lim_{x\to\infty}\left(\sqrt{x^2+2x}-x\right)$

$$= \lim_{x\to\infty}\left(\sqrt{x^2+2x}-x\right)\left(\frac{\sqrt{x^2+2x}+x}{\sqrt{x^2+2x}+x}\right)$$

$$= \lim_{x\to\infty}\frac{(\cancel{x^2}+2x-\cancel{x^2})}{\sqrt{x^2+2x}+x}$$

$$= \lim_{x\to\infty}\frac{2x}{\sqrt{x^2+2x}+x}$$

Looks like "$\infty-\infty$" but that's not defined.

We'll use the conjugate $\sqrt{x^2+2x}+x$ to help.

for $x>0$ $\sqrt{x^2}=|x|=x$

$$= \lim_{x \to \infty} \left(\frac{2x}{\sqrt{x^2+2x} + x} \right) \frac{\frac{1}{x}}{\frac{1}{x}}$$

Lets multiply by

$\frac{1}{x} / \frac{1}{x}$

$$= \lim_{x \to \infty} \frac{2}{\frac{1}{x}\sqrt{x^2+2x} + 1}$$

$$= \lim_{x \to \infty} \frac{2}{\frac{1}{\sqrt{x^2}}\sqrt{x^2+2x} + 1}$$

$$= \lim_{x \to \infty} \frac{2}{\sqrt{\frac{1}{x^2}(x^2+2x)} + 1} = \lim_{x \to \infty} \frac{2}{\sqrt{1 + \frac{2}{x}} + 1} = \frac{2}{\sqrt{1+0} + 1} = 1$$

Question

Evaluate if possible

$$\lim_{x\to\infty} \frac{\sqrt{3x^2+2x}}{4x+3}$$

(a) DNE

(b) $\frac{3}{4}$

(c) $\sqrt{3}$

(d) $\frac{\sqrt{3}}{4}$

$$= \lim_{x\to\infty} \frac{\sqrt{3x^2+2x}}{4x+3} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$$

use $\frac{1}{x} = \frac{1}{\sqrt{x^2}}$ for $x > 0$

$$= \lim_{x\to\infty} \frac{\sqrt{3x^2+2x}\left(\frac{1}{\sqrt{x^2}}\right)}{4+3/x}$$

$$= \lim_{x\to\infty} \frac{\sqrt{3+2/x}}{4+3/x} = \frac{\sqrt{3+0}}{4+0} = \frac{\sqrt{3}}{4}$$

Infinte Limits at Infinity

The following limits may arise

$$\lim_{x\to\infty} f(x) = \infty, \qquad \lim_{x\to\infty} f(x) = -\infty$$

$$\lim_{x\to-\infty} f(x) = \infty, \qquad \lim_{x\to-\infty} f(x) = -\infty$$

Two critical limits to remember (YOU'LL NEED TO KNOW THESE)

$$\lim_{x\to\infty} e^x = \infty \quad \text{and} \quad \lim_{x\to\infty} \ln(x) = \infty$$

Vertical and Horizontal Asymptotes

Vertical Asymptotes: The line $x = c$ is a *vertical asymptote* to the graph of f if

$$\lim_{x \to c^+} f(x) = \pm\infty, \quad \text{or} \quad \lim_{x \to c^-} f(x) = \pm\infty.$$

Horizontal Asymptotes: The line $y = L$ is a *horizontal asymptote* to the graph of f if

$$\lim_{x \to \infty} f(x) = L, \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L.$$

A good candidate for a vertical asymptote would be a number that makes a denominator zero.

Find any vertical and horizontal asymptotes to the graph of

$$f(x) = \frac{2x^2 - 4x - 6}{x^2 - 3x - 4}$$

Horizontal:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2x^2-4x-6}{x^2-3x-4}$$

$$= \lim_{x\to\infty} \left(\frac{2x^2-4x-6}{x^2-3x-4}\right) \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{2 - \frac{4}{x} - \frac{6}{x^2}}{1 - 3/x - 4/x^2} = \frac{2-0-0}{1-0-0} = 2$$

The line $y = 2$ is a hor. asympt.

$$\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} \frac{2x^2-4x-6}{x^2-3x-4} = 2$$

you fill in the details

Vertical: Find candidates, where is the denominator zero?

$$x^2-3x-4=0 \Rightarrow (x-4)(x+1)=0$$

$$x=4 \text{ or } x=-1$$

Check $c=4$

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \frac{2x^2-4x-6}{x^2-3x-4}$$

$$= \lim_{x\to 4^+} \frac{2x^2-4x-6}{(x-4)(x+1)} = \infty$$

Top goes to 10

$x+1 \to 5$

$x>4 \Rightarrow$

$x-4>0$

$$\frac{+}{(+)(+)}$$

$\boxed{\text{The line } x=4 \text{ is a vertical asymptote}}$

Check $x=-1$

$$\lim_{x\to -1} f(x) = \lim_{x\to -1} \frac{2x^2-4x-6}{(x-4)(x+1)}$$

$$= \lim_{x\to -1} \frac{2(x^2-2x-3)}{(x-4)(x+1)}$$

$$= \lim_{x\to -1} \frac{2(x-3)(x+1)}{(x-4)(x+1)} = \lim_{x\to -1} \frac{2(x-3)}{x-4} = \frac{2(-4)}{-5}$$

$$= \frac{8}{5}$$

$x=-1$ is NOT an asymptote

Questions

(1) **True or False:** Since $\lim_{x \to 0^+} \ln(x) = -\infty$, we can conclude that the line $x = 0$ is a vertical asymptote to the graph of $y = \ln(x)$. True

(2) **True or False:** Since $\lim_{x \to -\infty} e^x = 0$, we can conclude that the line $y = 0$ is a horizontal asymptote to the graph of $y = e^x$. True

Section 2.1: Rates of Change and the Derivative

We opened by saying that Calculus is concerned with the way in which quantities change. An obvious example of change is motion of an object in space (change of position).

Here we introduce the idea of *rate of change* and the mathematical formulation of this called a *derivative*.

Though we'll use **rectilinear motion** (i.e. movement along a straight line) as an illustrative example, the concept can be applied to many processes in physics, chemistry, biology, business, and the list goes on!

Motivational Example:

Suppose a ball is dropped from the top of the Space Needle 605 feet high. According to Galileo's law, the distance $s(t)$ feet the ball has fallen after t seconds is (neglecting wind drag)

$$s(t) = 16t^2.$$

The position of the ball relative to the top of the tower is changing. We can consider the ball's velocity.

We define **average velocity** as

$$\text{change in position} \quad \div \quad \text{change in time.} \qquad \frac{\Delta s}{\Delta t}$$

average velocity = change in position $\div$ change in time

Find the average velocity over the period from $t = 0$ to $t = 2$. seconds

$s(t) = 16t^2$

Avg. velocity $\frac{\Delta s}{\Delta t}$

$\Delta s = s(2) - s(0)$

$= 16(2^2) - 16(0^2) = 64 \text{ ft}$

$\Delta t = 2 - 0 = 2 \text{ sec}$

Avg velocity $= \frac{64 \text{ ft}}{2 \text{ sec}} = 32 \frac{\text{ft}}{\text{sec}}$

average velocity = change in position ÷ change in time

Find the average velocity over the period from $t = 2$ to $t = 4$. seconds

$$\Delta s = s(4) - s(2) = 16(4^2) - 16(2^2) = 256 - 64 \text{ ft}$$

$$= 192 \text{ ft}$$

$$\Delta t = 4 - 2 \text{ sec} = 2 \text{ sec}$$

$$\text{Avg. velocity} = \frac{192 \text{ ft}}{2 \text{ sec}} = 96 \frac{\text{ft}}{\text{sec}}$$

Here's a tougher question...

What is the *instantaneous velocity* when $t = 2$?

$\Delta t = 0$ so $\frac{\Delta s}{\Delta t}$ doesn't make sense.

Lets consider nonzero time intervals and take a limit. Let's consider the times $t=2$ and $t = 2 + \Delta t$ where $\Delta t \neq 0$.

The average velocity over this interval is

$$\frac{\Delta s}{\Delta t} = \frac{s(2+\Delta t) - s(2)}{2+\Delta t - 2} = \frac{s(2+\Delta t) - s(2)}{\Delta t}$$