

Sept. 9 Math 1190 sec. 51 Fall 2016

Section 2.1: Rates of Change and the Derivative

Suppose a ball is dropped from the top of the Space Needle 605 feet high. According to Galileo's law, the distance $s(t)$ feet the ball has fallen after t seconds is (neglecting wind drag)

$$s(t) = 16t^2.$$

The position of the ball relative to the top of the tower is changing. We can consider the ball's velocity.

We define **average velocity** as

$$\text{change in position} \quad \div \quad \text{change in time.}$$

Instantaneous Velocity

What is the *instantaneous velocity* when $t = 2$?

The time interval would be zero.

We'll let $\Delta t \neq 0$ and consider average velocity over time interval from $t = 2$ to $t = 2 + \Delta t$.

$$\begin{aligned} \text{avg. vel.} &= \frac{\Delta s}{\Delta t} = \frac{s(2 + \Delta t) - s(2)}{2 + \Delta t - 2} \\ &= \frac{16(2 + \Delta t)^2 - 16(2)^2}{\Delta t} \end{aligned}$$

Estimating instantaneous velocity using intervals of decreasing size...

Δt	$\frac{s(2+\Delta t)-s(2)}{\Delta t}$	Δt	$\frac{s(2+\Delta t)-s(2)}{\Delta t}$
1	80	-1	48
0.1	65.6	-0.1	62.4
0.05	64.8	-0.05	63.2
0.01	64.16	-0.01	63.84

The numbers appear to tend to 64.

In fact

$$\lim_{\Delta t \rightarrow 0} \frac{\Delta s}{\Delta t} = 64$$

at $t=2$

Instantaneous Velocity

If we consider the independent variable t and dependent variable $s = f(t)$, we note that the velocity has the form

$$\frac{\text{change in } s}{\text{change in } t} = \frac{\Delta s}{\Delta t}$$

Definition: We define the instantaneous velocity v (simply called *velocity*) at the time t_0 as

$$v = \lim_{\Delta t \rightarrow 0} \frac{\Delta s}{\Delta t} = \lim_{t \rightarrow t_0} \frac{f(t) - f(t_0)}{t - t_0}$$

provided this limit exists.

Example

An object moves along the x -axis such that its distance s from the origin at time t is given by $s = \sqrt{2t}$. If s is in inches and t is in seconds, determine the object's velocity at $t = 3$ sec.

$$\text{velocity} = \lim_{t \rightarrow t_0} \frac{f(t) - f(t_0)}{t - t_0} \quad \text{Here, } t_0 = 3 \text{ sec}$$
$$s = f(t) = \sqrt{2t}$$

$$= \lim_{t \rightarrow 3} \frac{\sqrt{2t} - \sqrt{2 \cdot 3}}{t - 3}$$

$$= \lim_{t \rightarrow 3} \frac{\sqrt{2t} - \sqrt{6}}{t - 3} \cdot \left(\frac{\sqrt{2t} + \sqrt{6}}{\sqrt{2t} + \sqrt{6}} \right) \quad \text{Use conjugate}$$

$$= \lim_{t \rightarrow 3} \frac{2t - 6}{(t-3)(\sqrt{2t} + \sqrt{6})}$$

$$= \lim_{t \rightarrow 3} \frac{2(t-3)}{(t-3)(\sqrt{2t} + \sqrt{6})}$$

$$= \lim_{t \rightarrow 3} \frac{2}{\sqrt{2t} + \sqrt{6}} = \frac{2}{\sqrt{2 \cdot 3} + \sqrt{6}} = \frac{2}{\sqrt{6} + \sqrt{6}}$$

$$= \frac{2}{2\sqrt{6}} = \frac{1}{\sqrt{6}}$$

The velocity of the object @ 3 seconds

is $\frac{1}{\sqrt{6}} \frac{\text{in}}{\text{sec.}}$

This is equivalent to $\frac{\sqrt{6}}{6} \frac{\text{in}}{\text{Sec}}$

Question

A cannon ball is fired from the ground so that its distance from the ground after t seconds is given by $s = 80t - 16t^2$ feet. Which of the following limits would be used to determine the ball's velocity at $t = 3$ seconds?

(a) $\lim_{t \rightarrow 0} \frac{80t - 16t^2 - 96}{t}$

(b) $\lim_{t \rightarrow 3} \frac{80t - 16t^2 - 96}{t - 3}$

(c) $\lim_{t \rightarrow 0} \frac{80t - 16t^2 - 96}{t - 3}$

(d) $\lim_{t \rightarrow 3} \frac{80t - 16t^2 - 96}{t}$

← should be $t-3$
limit should be as $t \rightarrow 3$

Observation

Note that the average velocity has the form $\frac{\Delta s}{\Delta t}$. This ratio (should) look familiar. If we think graphically, with $s = f(t)$

$$\frac{\Delta s}{\Delta t} = \frac{\text{rise}}{\text{run}} = \text{slope}$$

The Tangent Line Problem

Given a graph of a function $y = f(x)$:

A **secant** line is a line connecting two points $P = (x_0, y_0)$ and $Q = (x_1, y_1)$ on the graph. The slope of a secant line is

$$\frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

Recall that if $P = (c, f(c))$ and $Q = (x, f(x))$ are distinct points, we denoted the slope of the secant line

$$m_{sec} = \frac{f(x) - f(c)}{x - c}$$

We had defined the slope of the tangent line as

$$m_{tan} = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \quad \text{if this limit exists.}$$

Example

Find the slope of the line tangent to the graph of $y = \frac{1}{x}$ at the point $(-1, -1)$.

Here $f(x) = \frac{1}{x}$ and $c = -1$

$$m_{\text{tan}} = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$$

$$= \lim_{x \rightarrow -1} \frac{\frac{1}{x} - \frac{1}{-1}}{x - (-1)}$$

$$= \lim_{x \rightarrow -1} \frac{\frac{1}{x} + 1}{x + 1}$$

$$= \lim_{x \rightarrow -1} \frac{\frac{1}{x} + \frac{x}{x}}{x + 1}$$

$$= \lim_{x \rightarrow -1} \frac{\frac{1+x}{x}}{x+1}$$

$$= \lim_{x \rightarrow -1} \frac{1+x}{x} \cdot \left(\frac{1}{x+1} \right)$$

$$= \lim_{x \rightarrow -1} \frac{\cancel{1+x}}{x(\cancel{x+1})}$$

$$= \lim_{x \rightarrow -1} \frac{1}{x} = \frac{1}{-1} = -1$$

The slope $m_{\text{tan}} = -1$

Example Continued...

Find the equation of the line tangent to the graph of $y = \frac{1}{x}$ at the point $(-1, -1)$.

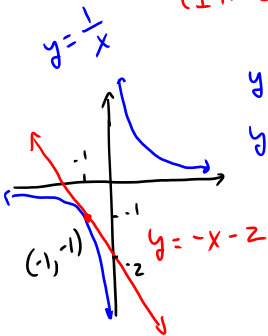
The point $(-1, -1)$ is given, and we found the slope of this line to be $m_{\text{tan}} = -1$.

(I'll use $y - y_0 = m(x - x_0)$.)

$$y - (-1) = -1(x - (-1))$$

$$y + 1 = -x - 1 \Rightarrow$$

$$y = -x - 2$$



Tangent Line

Theorem: Let $y = f(x)$ and let c be in the domain of f . If the slope m_{tan} exists at the point $(c, f(c))$, then the equation of the line tangent to the graph of f at this point is

$$y = m_{tan}(x - c) + f(c).$$

$$y - y_0 = m(x - x_0)$$

$$y - f(c) = m_{tan}(x - c)$$

The Derivative

Let $y = f(x)$. For $x \neq c$ we'll call $\frac{f(x) - f(c)}{x - c}$ the average rate of change of f on the interval from x to c .

We'll call

$$\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \quad \text{the rate of change of } f \text{ at } c$$

if this limit exists.

Definition: Let $y = f(x)$ at let c be in the domain of f . The **derivative** of f at c is denoted $f'(c)$ and is defined as

$$f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$$

provided the limit exists.

$f'(c)$ is read
"f prime of c"
or "at"