Sept. 9 Math 1190 sec. 52 Fall 2016

Section 2.1: Rates of Change and the Derivative

Suppose a ball is dropped from the top of the Space Needle 605 feet high. According to Galileo's law, the distance $s(t)$ feet the ball has fallen after t seconds is (neglecting wind drag)

$$s(t) = 16t^2.$$

The position of the ball relative to the top of the tower is changing. We can consider the ball's velocity.

We define **average velocity** as

$$\text{change in position} \quad \div \quad \text{change in time.}$$

Instantaneous Velocity

What is the *instantaneous velocity* when $t = 2$?

The time interval has length zero.

We'll consider the average velocity over a time interval from $t=2$ to $t=2+\Delta t$ for $\Delta t \neq 0$.

$$\text{avg. vel} = \frac{\Delta s}{\Delta t} = \frac{s(2+\Delta t) - s(2)}{2+\Delta t - 2}$$

$$= \frac{16(2+\Delta t)^2 - 16(2)^2}{\Delta t}$$

we can consider Δt "very small".

Estimating instantaneous velocity using intervals of decreasing size...

Δt	$\frac{s(2+\Delta t)-s(2)}{\Delta t}$	Δt	$\frac{s(2+\Delta t)-s(2)}{\Delta t}$
1	80	-1	48
0.1	65.6	-0.1	62.4
0.05	64.8	-0.05	63.2
0.01	64.16	-0.01	63.84

Velocities appear to be getting close to $64 \frac{ft}{sec}$

In fact

$$\lim_{\Delta t \to 0} \frac{s(2+\Delta t) - s(2)}{\Delta t}$$

$$= 64$$

Instantaneous Velocity

If we consider the independent variable t and dependent variable $s = f(t)$, we note that the velocity has the form

$$\frac{\text{change in } s}{\text{change in } t} \quad = \quad \frac{\Delta s}{\Delta t}$$

Definition: We define the instantaneous velocity v (simply called *velocity*) at the time t_0 as

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0}$$

provided this limit exists.

Example

An object moves along the x-axis such that its distance s from the origin at time t is given by $s = \sqrt{2t}$. If s is in inches and t is in seconds, determine the object's velocity at $t = 3$ sec.

$$\text{velocity} = \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0}$$, Here, $t_0 = 3$ and $f(t) = \sqrt{2t}$

$$= \lim_{t \to 3} \frac{\sqrt{2t} - \sqrt{2 \cdot 3}}{t - 3}$$

$$= \lim_{t \to 3} \frac{\sqrt{2t} - \sqrt{6}}{t - 3}$$

we'll use the conjugate

$$= \lim_{t \to 3} \left(\frac{\sqrt{2t} - \sqrt{6}}{t - 3} \right) \cdot \left(\frac{\sqrt{2t} + \sqrt{6}}{\sqrt{2t} + \sqrt{6}} \right)$$

$$= \lim_{t \to 3} \frac{2t - 6}{(t-3)(\sqrt{2t} + \sqrt{6})}$$

$$= \lim_{t \to 3} \frac{2\cancel{(t-3)}}{\cancel{(t-3)}(\sqrt{2t} + \sqrt{6})}$$

$$= \lim_{t \to 3} \frac{2}{\sqrt{2t} + \sqrt{6}} = \frac{2}{\sqrt{2 \cdot 3} + \sqrt{6}}$$

$$= \frac{2}{\sqrt{6} + \sqrt{6}} = \frac{2}{2\sqrt{6}} = \frac{1}{\sqrt{6}}$$

The velocity @ $t=3$ seconds is $\frac{1}{\sqrt{6}} \frac{\text{in}}{\text{sec}}$.

Question

A cannon ball is fired from the ground so that it's distance from the ground after t seconds is given by $s = 80t - 16t^2$ feet. Which of the following limits would be used to determine the ball's velocity at $t = 3$ seconds?

(a) $\lim_{t\to 0} \dfrac{80t - 16t^2 - 96}{t}$

(b) $\lim_{t\to 3} \dfrac{80t - 16t^2 - 96}{t-3}$

(c) $\lim_{t\to 0} \dfrac{80t - 16t^2 - 96}{t-3}$

(d) $\lim_{t\to 3} \dfrac{80t - 16t^2 - 96}{t}$

$\lim_{t\to t_0} \dfrac{f(t) - f(t_0)}{t - t_0}$

Observation

Note that the average velocity has the form $\frac{\Delta s}{\Delta t}$. This ratio (should) look familiar. If we think graphically, with $s = f(t)$

$$\frac{\Delta s}{\Delta t} = \frac{\text{rise}}{\text{run}} = \text{slope}$$

The Tangent Line Problem

Given a graph of a function $y = f(x)$:

A **secant** line is a line connecting two points $P = (x_0, y_0)$ and $Q = (x_1, y_1)$ on the graph. The slope of a secant line is

$$\frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

Recall that if $P = (c, f(c))$ and $Q = (x, f(x))$ are distinct points, we denoted the slope of the secant line

$$m_{sec} = \frac{f(x) - f(c)}{x - c}$$

We had defined the slope of the tangent line as

$$m_{tan} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \quad \text{if this limit exists.}$$

Example

Find the slope of the line tangent to the graph of $y = \frac{1}{x}$ at the point $(-1, -1)$.

$$m_{tan} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Here $f(x) = \frac{1}{x}$ and

$c = -1$.

Note $f(-1) = \frac{1}{-1} = -1$

$$= \lim_{x \to -1} \frac{\frac{1}{x} - \frac{1}{-1}}{x - (-1)}$$

$$= \lim_{x \to -1} \frac{\frac{1}{x} + 1}{x+1}$$

$$= \lim_{x \to -1} \frac{\frac{1}{x} + \frac{x}{x}}{x+1}$$

$$= \lim_{x \to -1} \frac{\frac{1+x}{x}}{x+1}$$

$$= \lim_{x \to -1} \frac{1+x}{x} \cdot \left(\frac{1}{x+1} \right)$$

$$= \lim_{x \to -1} \frac{\cancel{1+x}}{x(\cancel{x+1})}$$

$$= \lim_{x \to -1} \frac{1}{x} = \frac{1}{-1} = -1$$

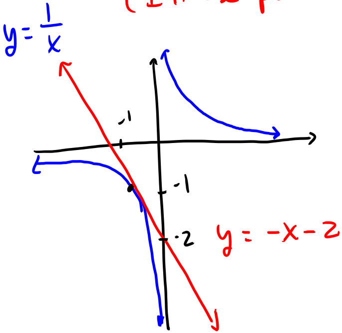
$$m_{tan} = -1 \quad @ \quad (-1, -1)$$

Example Continued...

Find the equation of the line tangent to the graph of $y = \frac{1}{x}$ at the point $(-1, -1)$.

The point is given as $(-1,-1)$. We found the slope $m_{tan} = -1$.

(I'll use point-slope form $y - y_0 = m(x - x_0)$.)



$y - (-1) = -1\,(x - (-1))$

$y + 1 = -x - 1 \implies \boxed{y = -x - 2}$

Tangent Line

Theorem: Let $y = f(x)$ and let c be in the domain of f. If the slope m_{tan} exists at the point $(c, f(c))$, then the equation of the line tangent to the graph of f at this point is

$$y = m_{tan}(x - c) + f(c).$$

$$y - y_0 = m(x - x_0)$$

$$y - f(c) = m_{tan}(x - c)$$

The Derivative

Let $y = f(x)$. For $x \neq c$ we'll call $\frac{f(x)-f(c)}{x-c}$ the average rate of change of f on the interval from x to c.

We'll call

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \quad \text{the rate of change of } f \text{ at } c$$

if this limit exists.

Definition: Let $y = f(x)$ at let c be in the domain of f. The **derivative** of f at c is denoted $f'(c)$ and is defined as

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

provided the limit exists.

The Derivative

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

In addition to *the derivative of f at c*, the notation $f'(c)$ is read as

- f prime of c, or
- f prime at c.

At this point, we have several interpretations of this same **number** $f'(c)$.

- as a velocity if f is the position of a moving object,
- as a rate of change of the function f when $x = c$,
- as the slope of the line tangent to the graph of f at $(c, f(c))$.