September 9 Math 2306 sec 51 Fall 2015

Section 3.1 (1.3, and a peek at 3.2) Applications

The equation

$$\frac{dP}{dt} = kP(M - P), \quad k, M > 0$$

is called a **logistic growth equation**.

Solve this equationand show that for any $P(0) = P_0 > 0$, $P \to M$ as $t \to \infty$.

We separated the variables and used the partial fraction decomposition $\frac{1}{P(M-P)} = \frac{1}{M}\left(\frac{1}{P} + \frac{1}{M-P}\right)$ to conclude that

$$\ln|P| - \ln|M - P| = Mkt + C$$

$$\ln\left|\frac{P}{M-P}\right| = Mkt + C$$

$$e^{\ln\left|\frac{P}{M-P}\right|} = e^{Mkt+C}$$

$$\left|\frac{P}{M-P}\right| = e^C e^{Mkt} \qquad \text{let } A = \pm e^C \text{ or } 0$$

$$\frac{P}{M-P} = A e^{Mkt} \qquad \Rightarrow \qquad P = A e^{Mkt}(M-P)$$

$$\Rightarrow \quad P(1 + Ae^{Mkt}) = AMe^{Mkt}$$

$$P = \frac{AM e^{Mkt}}{1 + Ae^{Mkt}} \qquad \text{Set } P(0) = P_0$$
$$\text{to find } A$$

$$P(0) = \frac{AM e^0}{1 + A e^0} = P_0 \implies \frac{AM}{1 + A} = P_0$$

$$AM = P_0(1 + A) \implies A(M - P_0) = P_0$$

$$A = \frac{P_0}{M - P_0}$$

$$P = \frac{\frac{P_0}{M - P_0} M e^{Mkt}}{1 + \frac{P_0}{M - P_0} e^{Mkt}} \cdot \frac{M - P_0}{M - P_0}$$

$$P = \frac{P_0 M e^{Mkt}}{M - P_0 + P_0 e^{Mkt}}$$

Take $t \to \infty$

$$\lim_{t \to \infty} P = \lim_{t \to \infty} \frac{P_0 M e^{Mkt}}{M - P_0 + P_0 e^{Mkt}} = " \frac{\infty}{\infty} "$$

Use l'Hopital's rule

$$= \lim_{t \to \infty} \frac{P_0 M (Mk) e^{Mkt}}{P_0 (Mk) e^{Mkt}}$$

$$= \lim_{t \to \infty} M = M$$

So $p(t) \to M$ for any nonzero initial population.

Original Equation

$$\frac{dP}{dt} = kP(M-P) \qquad k, M > 0$$

$P(0) = P_0 > 0$ (negative population doesn't make sense, so $P \geq 0$.)

- If $P > M$ $\frac{dP}{dt} < 0$ population decreases toward M

- If $P < M$, $\frac{dP}{dt} > 0$ population increases toward M